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Symmetry-adapted polynomial basis for global potential energy surfaces-applications to XY_4 molecules

Patrick Cassam-Chenaï · Frédéric Patras

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Abstract The nuclear potential energy surfaces (PES) used in quantum chemistry inherit the symmetry of the whole molecular system, and are therefore invariant under the action of the nuclear permutation-translation-rotation-inversion group. One can take advantage of this property, both theoretically and numerically. The present article is the prolongation of the works of Schmelzer and Murrell and of Collins and Parsons on the subject. It presents a simplified technique to obtain symmetry-adapted polynomial basis for global PES, together with algorithmic recipes that make the problem computationally tractable. The method is illustrated in detail on XY_4 type of molecules for which a full description of the algebra of invariant polynomials under the full symmetry group of the molecule is given.

1 Introduction

To study the outcome of chemical reactions, in particular when deep, geometrical changes occur, or to interprete the spectroscopy of floppy (possibly highly excited), ro-vibrational states, it is often necessary to deal with global potential energy surfaces (PES) for the nuclei or at least PES accurate in a large domain of the nuclear configuration space. However, various problems when determining global PES have been identified in the past.

First of all, it is, usually, relatively easy to construct a set of internal coordinates describing a molecule in the neighborhood of a given nuclear configuration [14].

P. Cassam-Chenaï (✉) · F. Patras
CNRS UMR 6621, Université de Nice, Laboratoire de Mathématiques J. Dieudonné, Parc Valrose, 06108 Nice cedex 2, France
e-mail: cassam@math.unice.fr

F. Patras
e-mail: patras@math.unice.fr

However, it is well-known that, in general, it is impossible to describe the whole of the PES with such a simple parametrization. To speak the language of differential geometry, more than one chart is required to construct an atlas for the PE manifold.

A second class of problems is related to the fact that the PES dealt with in quantum chemistry, inherit the symmetries of the molecule, in the sense that the action of any operator of the permutation-translation-rotation-inversion group (the group of operators commuting with the total Hamiltonian of the molecule) leaves the PES invariant. A computation that would lead to a non-invariant PES (e.g. by fitting the surface with a non-invariant analytic expression) would not be satisfactory from the theoretical point of view (and, in some cases, the expressions would not be as compact as they could be). However, the local coordinates used in quantum chemistry are not always symmetry-adapted. For instance, the widely used "normal coordinates" are not symmetry-adapted, when the geometry of the molecule has not the maximal symmetry allowed by the symmetry point group (think of the ozone molecule). Or, the reaction coordinates used to describe the dissociation of an atom, seldom respect the symmetry, when the molecule contain several indistinguishable such atoms (think of the dissociation of methane).

The purpose of the present article, is to present a simplified technique, based on the theory of algebraic invariants to obtain symmetry-adapted parameters for global PES. The method is illustrated on XY_4-type of molecule.

The introduction of the theory of algebraic invariants to expand a PES can be traced back to the work of Schmelzer and Murrell [10]. In their approach, they used the Molien series associated to finite molecular point group actions to obtain the dimensions of basis made of homogeneous invariant polynomials of internal coordinates. Since, given a linear representation of a (finite or, more generally, compact) group G, all smooth G-invariant functions are smooth functions of invariant polynomials (Schwarz 1975, see e.g. [9]), this approach is suitable to express any polynomial, analytic or C^∞ invariant functions.

The Schmelzer–Murrell approach was extended by Collins and Parsons [5] to cover the rotation-inversion group action. In particular, they represented the molecular symmetry group on a set of $O(3)$-invariant internal coordinates ($O(3)$ denoting the orthogonal group of a dimension 3 Euclidean space) and applied the Molien series treatment to the action of the full permutation-rotation-inversion group. They also addressed the problem of the indeterminacy of certain nuclear configurations for molecules of 5 nuclei or more, which can only be removed with the addition of redundant symmetry-adapted coordinates. Our article should be understood as a prolongation of their work, the applications of which were restricted to molecules with small nuclear permutation groups in practice.

Our global approach to the problem is different from theirs, by the fact that we will treat separately and successively the action of the orthogonal group and that of the group of indistinguishable nuclei permutations. This simple idea makes amenable the powerful algorithms of invariant theory for finite groups. So, although we have chosen the relatively small penta-atomic molecule as an illustration, the study of bigger molecules becomes computationally more tractable.

We have detailed a simple algorithmic procedure for constructing an integrity basis (i.e. a complete family of algebraic generators of the algebra of polynomial invariants adapted to the structure of the corresponding Molien series, see below for a precise definition). Various already existing algorithms could theoretically be used for the same purpose, such as those associated to Gröbner basis computations (see e.g. [6]). However, for intrinsic complexity reasons, they do not seem to be able to treat high-dimensional problems such as those encountered in molecular PES computations.

The article is organized as follows. In the next section, we recall some preliminary results on internal coordinates invariant under the action of $O(3)$ and introduce various notations. In the third section, we explain how a complete system of invariant polynomials can be constructed recursively for XY_4 molecules. The resulting minimal generating family of symmetry-adapted functions is listed in Sect. 3.2 and the Appendix. In conclusion, we emphasize the points of our approach which are general and those that are specific to the example chosen as an illustration.

2 Symmetry-adaptation to the rotation-inversion group, $O(3)$

In quantum chemical calculations, molecular geometries are often described by the so-called Z-matrix in terms of internal coordinates such as bond lengths, bond angles and dihedral angles. However, if a minimal set ($3N - 5$ or $3N - 6$ depending on the molecule being linear or not) of such internal coordinates can always be defined locally, it is not so globally, where many problems occur. For example, dihedral angles are only defined at the nuclear configurations where the three nuclei used to determine a reference plane are not aligned. If one avoids such "out-of-plane" angles and try to use a locally complete set of $3N - 6$ "in-plane" internal coordinates (i.e. internuclear distances and in-plane angles, we do not consider the configurations where two nuclei coalesce), then for molecules containing five atoms or more, the same values taken by the set of $3N - 6$ coordinates can correspond to several physically inequivalent geometries [5,12].

This phenomenom means that a locally complete coordinate systems is not globally complete. So, one or several, "redundant" coordinates have to be added to the "basic" ones in order to get a one-to-one correspondance between sets of coordinates and molecular shapes. Of course, for dimensional reasons, the redundant variables cannot vary freely: as a matter of fact, the PES is a $(3N - 6)$-dimensional manifold (except in a domain of measure zero), so the redundant coordinates must be constrained to satisfy algebraic equations (called syzygies in the language of ring theory) relating them to the $(3N - 6)$ basic, free coordinates.

The fundamental reason for this need of redundant coordinates is explained by a classical result due to Weyl. See also Collins and Parson [5] for an argument centered on a generalization of Molien theorem to $O(3)$. Recall here, that rather than considering the case of general, invariant, smooth functions we can limit the study to the case of invariant polynomials in the Cartesians coordinates. Weyl's second main theorem for the orthogonal group [13] shows that the natural choice of, possibly redundant, $O(3)$-invariant coordinates, is the set of the $\frac{N(N-1)}{2}$ scalar products of $(N - 1)$ "internal" vectors, and that there is no linear syzygy relating these $O(3)$-invariant polynomials, since any possible syzygy is generated by polynomials of degree at least 4 in these scalar product coordinates. So, the algebra of $O(3)$-invariant polynomials in the Cartesians coordinates, $\mathcal{P}$, is the algebra spanned by exactly $\frac{N(N-1)}{2}$ polynomial invariant of degree 2. Out of this minimal set of $\frac{N(N-1)}{2}$ generators, one is free to form (by linear combination) $(3N-6)$ basic, algebraically independent coordinates and $\frac{N(N-1)}{2}-(3N-6)=\frac{N^2-7N+12}{2}$ auxiliary invariant coordinates, related to the basic ones through the syzygies.

To fix the ideas and for later applications, let us consider the methane molecule. For this molecule, we will take for body-fixed origin the Radau origin X defined as follows. Let B_H be the barycenter of the hydrogen atoms H_i, $i=1,\ldots,4$, B the total barycenter of the molecule and C the carbon atom. Then, X is given by the geometrical mean,

$$\overline{B_H X}^2 = \overline{B_H C}\cdot\overline{B_H B} \tag{1}$$

The shape of the molecule is entirely determined by the 12 Cartesian coordinates of the four Radau vectors corresponding to hydrogen nuclei positions, $\overrightarrow{r}_i = (x_i^1, x_i^2, x_i^3)$, $i=1,\ldots,4$, where:

$$\overrightarrow{r_i} = \overrightarrow{XH_i}. \tag{2}$$

Since the PES is invariant under the action of isometries, and since the topological dimension of $O(3)$ is 3, locally, a system of $9=12-3$ coordinates is enough to describe the shape of the methane molecule. However, in order to parametrize the PES globally, we follow Weyl and define 10 $O(3)$-invariant coordinates by setting,

$$d_{ij} := \langle\overrightarrow{r_i}\,|\,\overrightarrow{r_j}\rangle. \tag{3}$$

We further ask that the coordinates be adapted to the $\mathcal{S}_4$ nuclear permutation symmetry of the molecule, and choose the following 9 basic coordinates:

$$S_1 := \frac{1}{2}(d_{11}+d_{22}+d_{33}+d_{44}) \tag{4}$$

$$S_{2a} := \frac{1}{\sqrt{12}}(2d_{12}-d_{13}-d_{14}-d_{23}-d_{24}+2d_{34}) \tag{5}$$

$$S_{2b} := \frac{1}{2}(d_{13}-d_{14}-d_{23}+d_{24}) \tag{6}$$

$$S_{3x} := \frac{1}{2}(d_{11}-d_{22}+d_{33}-d_{44}) \tag{7}$$

$$S_{3y} := \frac{1}{2}(d_{11}-d_{22}-d_{33}+d_{44}) \tag{8}$$

$$S_{3z} := \frac{1}{2}(d_{11}+d_{22}-d_{33}-d_{44}) \tag{9}$$

$$S_{4x} := \frac{1}{\sqrt{2}}(d_{24}-d_{13}) \tag{10}$$

$$S_{4y} := \frac{1}{\sqrt{2}}(d_{23} - d_{14}) \tag{11}$$

$$S_{4z} := \frac{1}{\sqrt{2}}(d_{34} - d_{12}). \tag{12}$$

These are essentially the usual T_d-symmetry-adapted linear combinations used in many studies on XY_4 molecules, but here, they are linear combinations of the d_{ij} instead of bond lengths and bond angles or cosines of bond angles. These new coordinates are degree 2 polynomials in the Cartesians, which generate a subalgebra of the polynomial algebra over the Cartesian coordinates. The system is locally complete, for example at the neighborhood of the equilibrium geometry: in this neighborhood, the molecular shapes are described without ambiguity by these nine coordinates.

However, as already mentioned, the system is not complete [5,12]. That is, the knowledge of the nine coordinates does not always determine the shape of the molecule unambiguously. In such a situation, as a consequence of Weyl's result, one has to introduce an extra (or *redundant*) symmetry coordinate, for example,

$$S_5 := \frac{1}{\sqrt{6}}(d_{12} + d_{13} + d_{14} + d_{23} + d_{24} + d_{34}), \tag{13}$$

to make the system complete, S_5 being the solution of a unique, monic, quartic, polynomial syzygy:

$$X^4 + \alpha_3 X^3 + \alpha_2 X^2 + \alpha_1 X + \alpha_0,$$

where α_i is a homogeneous polynomial of degree $4 - i$ into the remaining coordinates $S_1, \ldots, S_{4z}$. This property implies that any $O(3)$-invariant polynomial P in the Cartesians can be written uniquely as:

$$P = P_0 + P_1 S_5 + P_2 S_5^2 + P_3 S_5^3, \tag{14}$$

where the P_i are polynomials in the algebraically free variables $S_1, \ldots, S_{4z}$.

3 Symmetry-adaptation to the nuclear permutation-rotation-inversion group, $O(3) \times G$

Since a PES is invariant under the nuclear symmetry group obtained as the product of the group of affine isometries (the semi-direct product of $O(3)$ with the translations) and the nuclei permutation group, denoted G, it follows from the Schwarz result recalled in the introduction that any smooth, analytic or polynomial expression of the PES can be obtained in terms of $O(3) \times G$-invariant polynomials. Our ultimate goal, is to generate in the most economical way the algebra of these polynomials.

In the previous section, we have seen how to generate the algebra of $O(3)$-invariant polynomials. In particular, we have found that, for XY_4 molecules, a general $O(3)$-invariant polynomial has the form of Eq. 14. In the present section, we rely on a fundamental result of ring theory stating that $O(3) \times G$-invariant polynomials have a general decomposition similar to Eq. 14.

The theoretical framework to describe invariants in polynomial algebras under finite group actions is well known, both to mathematicians and chemists, so, we will drop details and focus on the main results. Recall however, once again, the pioneering influence of Schmelzer and Murrell [10], as far as the determination of PES is concerned. Besides [10], classical references on the subject are Refs. [1,3,11] or, Ref. [9] for an overview of the various possible applications to chemistry and physics.

3.1 Hironaka decomposition

Let $\mathcal{P}$ denote the ring of $O(3)$-invariant polynomials in the Cartesians coordinates for the field of real numbers, $\mathbb{R}$, constructed in the previous section, $\mathcal{P} = \bigoplus_{n\geq 0} \mathcal{P}_{2n}$, where $\mathcal{P}_i$ is the vector space of $O(3)$-invariant polynomials of degree i, and let $\mathcal{P}^G$ be the subalgebra of polynomials also invariant under a linear action of group G. One see easily that $\mathcal{P}^G = \bigoplus_{n\geq 0} \mathcal{P}^G_{2n}$, where $\mathcal{P}^G_i = \mathcal{P}^G \cap \mathcal{P}_i$ is the vector space of $O(3) \times G$-invariant polynomials of degree i.

Note that $O(3)$ is a reductive group that is to say (see p. 284 of [2]) it has no connected, unipotent, Abelian, normal subgroup other than $\{Id\}$. Recall that a unipotent matrix group, (G,+), is a matrix group made of unipotent elements i.e. elements that are the sum of the identity and a nilpotent element, $\forall g \in G,\ \exists f \in G,\ n \in \mathbb{N},$ *such that* $g = Id + f$, *and* $f^n = 0$. The property is clear here because, up to conjugacy in $GL(3)$, no element of $O(3)$ can be represented by an upper triangular matrix with all diagonal elements equal to one, except the identity.

So, $O(3)$ being a reductive group and $\mathbb{R}$ being of characteristic 0, the main theorem of Hochster and Roberts ([8], p. 115) applies to it and also extends to $O(3) \times G$ (see [8] p. 153 and [7] p. 482): for any maximal set of homogeneous, algebraically independent, invariant polynomials, $f_1, \ldots, f_m$ ($m = 3N - 6$ in the case of a PES), $\mathcal{P}^G$ is a finitely generated, free module over the algebra, $\mathbb{R}[f_1, \ldots, f_m]$, spanned by $f_1, \ldots, f_m$, i.e. there exist a finite number of invariant polynomials, $g_1, \ldots, g_p$, such that

$$\mathcal{P}^G = \mathbb{R}[f_1, \ldots, f_m] \oplus \mathbb{R}[f_1, \ldots, f_m]g_1 \oplus \cdots \oplus \mathbb{R}[f_1, \ldots, f_m]g_p. \tag{15}$$

Such a decomposition is sometimes referred to as a "Hironaka decomposition", and defines a so-called "Cohen-Macaulay" algebra. The Eq. 14 is a concrete example of such a decomposition. The f_i are called the basic invariants, and the g_j the auxiliary invariants. The whole set $\{f_1, \ldots, f_m; g_1, \ldots, g_p\}$ (where we put a semicolon ";" to separate the two types of invariants) is referred to as an integrity basis. Note that contrary to the $O(3)$ case dealt with in the previous section, there is no reason why the auxiliary invariants of higher degrees could be expressed as product of auxiliary invariants of lesser degrees (degree 2 in the case of $O(3)$).

The elements of an integrity basis can always be chosen homogeneous, and from now on, we will always assume that the homogeneity property holds. Notice that, even with this assumption, there is no unicity in the above construction: for example, if $G = \{Id\}$ acts on the polynomial algebra $\mathbb{R}[x]$, $\{x\}$ and $\{x^2; x\}$ are two integrity basis. However, for a given choice of the basic invariants, the number of the auxiliary invariants and their degrees are fixed and determined by the so-called "Molien series", as we shall see now.

By definition, the Molien series, $Mol(t)$, associated to the representation of $O(3) \times G$ on $\mathcal{P}$ we are interested in, is the Hilbert series (also called Poincaré series), $Hilb(\mathcal{P}^G, t)$, of the graded algebra $\mathcal{P}^G$,

$$Hilb(\mathcal{P}^G, t) = \sum_{i \geq 0} dim\mathcal{P}_i^G \, t^i, \tag{16}$$

up to the normalisation factor, $\frac{1}{Card(G)}$. Suppose that $\{f_1, \ldots, f_m; g_1, \ldots, g_p\}$ is a given integrity basis, then the Molien series can be cast in the following form,

$$Mol(t) = \frac{1 + t^{deg(g_1)} + \cdots + t^{deg(g_p)}}{(1 - t^{deg(f_1)}) \ldots (1 - t^{deg(f_m)})}. \tag{17}$$

So, if the degrees of the basic invariants are given, then the quantity, $Mol(t) \cdot (1 - t^{deg(f_1)}) \ldots (1 - t^{deg(f_m)})$ determines the number of auxiliary invariants of each degree (note that the degrees are not necessarily distinct in this expression). The problem of generating $\mathcal{P}^G$ comes down to the computation of a complete set of such auxiliary invariants given a set of basic invariants.

Let us return to the CH_2 example with the coordinate system of Sect. 1. The nuclei permutation group G is the symmetric group of order 4, $\mathcal{S}_4$. We only need to consider the reduced representation, $\mathcal{P}'_2$, spanned by $\left(S_{2a}, S_{2b}, S_{3x}, S_{3y}, S_{3z}, S_{4x}, S_{4y}, S_{4z}\right)$, since S_1 and S_5 transform as the trivial representation of $\mathcal{S}_4$. In other terms, the computation of the invariant polynomials under the full symmetry group of the molecule reduces to the determination of $\mathcal{S}_4$-invariants polynomials in the subalgebra $\mathcal{P}'$ generated by $S_{2a}, \ldots, S_{4z}$ of the polynomial algebra over the cartesians. Since $S_{2a}, \ldots, S_{4z}$ is a family of algebraically independent variables, $\mathcal{P}'$ is (isomorphic to) the polynomial algebra $\mathbb{R}[S_{2a}, \ldots, S_{4z}]$, and the action of $\mathcal{S}_4$ on this algebra is induced by a linear representation of $\mathcal{S}_4$ on the linear span of $S_{2a}, \ldots, S_{4z}$.

The Molien series for the action of $\mathcal{S}_4$ on $\mathcal{P}'$ can be computed explicitly and reads:

$$Mol(t) = \frac{1 + t^2 + 5t^3 + 9t^4 + 12t^5 + 18t^6 + 21t^7 + 24t^8 + 26t^9 + 15t^{10} + 8t^{11} + 4t^{12}}{(1 - t^2)^3 (1 - t^3)^3 (1 - t^4)^2} \tag{18}$$

Since, as we shall see, there exists a set of 8 basic invariants corresponding to the denominator (i.e. such that three of them are of degree two, three others are of degree three, the remaining two being of degree four), the value of the numerator minus one for $t = 1$ shows that there are 143 auxiliary invariants to determine. So, if we denote by $\{f_1, \ldots, f_8; g_1, \ldots, g_{143}\}$ an integrity basis corresponding to the Molien series, Eq. 18, a general $O(3) \times \mathcal{S}_4$-invariant polynomial element of $\mathcal{P}^G$ will identify with a unique linear combination of monomials:

$$S_1^i f_1^{j_1} \dots f_8^{j_8} g_k S_5^l \quad i, j_1, \dots, j_8 \in \mathbb{N}, \quad 0 \le k \le 143, \ 0 \le l \le 3, \tag{19}$$

where we set $g_0 := 1$.

3.2 Algebrically independent generators

Let us construct first the family $f_1, \dots, f_8$ of basic invariants.

Notice that the representation of $\mathcal{S}_4$ on the vector space $\mathbb{R}\langle S_{2a}, \dots, S_{4z}\rangle$ generated by $S_{2a}, \dots, S_{4z}$ splits into a direct sum of irreducible representations:

$$\mathbb{R}\langle S_{2a}, S_{2b}\rangle \oplus \mathbb{R}\langle S_{3x}, S_{3y}, S_{3z}\rangle \oplus \mathbb{R}\langle S_{4x}, S_{4y}, S_{4z}\rangle. \tag{20}$$

To each of these irreducible representations is associated a polynomial algebra ($\mathbb{R}[S_{2a}, S_{2b}]$, $\mathbb{R}[S_{3x}, S_{3y}, S_{3z}]$, and $\mathbb{R}[S_{4x}, S_{4y}, S_{4z}]$ respectively), and a module of invariant polynomials ($\mathbb{R}[S_{2a}, S_{2b}]^{\mathcal{S}_4}$, $\mathbb{R}[S_{3x}, S_{3y}, S_{3z}]^{\mathcal{S}_4}$, and $\mathbb{R}[S_{4x}, S_{4y}, S_{4z}]^{\mathcal{S}_4}$ respectively). These modules are in fact polynomial algebras, that is, for a suitable choice of the basic invariants, no auxiliary invariant appear in their Hironaka decomposition.

This follows from Theorem 7.2.1 p. 83 of [1] (see also [4]). This result gives for the case of interest here:

Theorem 1 *Let V be an n-dimensional vector space and G be a finite subgroup of $GL(V)$. There exist n algebraically independent homogeneous invariants $f_1, \dots, f_n$ such that $\mathbb{R}[V]^G = \mathbb{R}[f_1, \dots, f_n]$ if and only if G is generated by pseudoreflections, that is by linear transformations of finite order whose fixed space has dimension $n-1$.*

Although the representation of $\mathcal{S}_4$ on the vector space $\mathbb{R}\langle S_{2a}, \dots, S_{4z}\rangle$ is not generated by pseudoreflections, it can be shown that the subrepresentation of $\mathcal{S}_4$ on $\mathbb{R}\langle S_{2a}, S_{2b}\rangle$, $\mathbb{R}\langle S_{3x}, S_{3y}, S_{3z}\rangle$, and $\mathbb{R}\langle S_{4x}, S_{4y}, S_{4z}\rangle$ respectively, are generated by pseudoreflections and therefore, Theorem 1 applies.

Consequently, the search for the 8 basic invariants of the initial representation reduces to the search of the 2 basic invariants of $\mathbb{R}[S_{2a}, S_{2b}]^{\mathcal{S}_4}$, the 3 basic invariants of $\mathbb{R}[S_{3x}, S_{3y}, S_{3z}]^{\mathcal{S}_4}$ and of 3 basic invariants of $\mathbb{R}[S_{4x}, S_{4y}, S_{4z}]^{\mathcal{S}_4}$.

In fact, their forms are familiar as they have already appeared in the literature. We list them below by degrees of increasing order:

(1) Degree 2:

$$f_1 = I_2^2 := \frac{S_{2a}^2 + S_{2b}^2}{\sqrt{2}} \tag{21}$$

$$f_2 = I_3^2 := \frac{S_{3x}^2 + S_{3y}^2 + S_{3z}^2}{\sqrt{3}} \tag{22}$$

$$f_3 = I_4^2 := \frac{S_{4x}^2 + S_{4y}^2 + S_{4z}^2}{\sqrt{3}} \tag{23}$$

(2) Degree 3:

$$f_4 = I_2^3 := \frac{S_{2a}^3 - 3S_{2b}^2 S_{2a}}{\sqrt{10}} \quad (24)$$

$$f_5 = I_3^3 := S_{3x} S_{3y} S_{3z} \quad (25)$$

$$f_6 = I_4^3 := S_{4x} S_{4y} S_{4z} \quad (26)$$

(3) Degree 4:

$$f_7 = I_3^4 := \frac{S_{3x}^4 + S_{3y}^4 + S_{3z}^4}{\sqrt{3}} \quad (27)$$

$$f_8 = I_4^4 := \frac{S_{4x}^4 + S_{4y}^4 + S_{4z}^4}{\sqrt{3}}. \quad (28)$$

3.3 Algorithm for the determination of auxilary invariants

The determination of the auxiliary invariants is done inductively, using the Molien series and the family of basic invariants already obtained. We detail the inductive process, since the construction could be generalized to the determination of invariant polynomials for any molecule and any nuclear symmetry group.

Notice first that one can take advantage of the algebra structure of the representation ring of G to refine the Molien series computation -and split the series into pieces, see e.g. [9]. This property has a simple counterpart when computing auxiliary invariants. Namely, the algebra of invariant polynomials under the action of the symmetry group of the nuclei splits into multi-graded pieces.

Concretely, it can be shown easily that one can choose auxiliary invariants of the algebra of polynomial invariants that are homogeneous when considered as polynomials over any of the set of variables: $\{S_{2a}, S_{2b}\}$, $\{S_{3x}, S_{3y}, S_{3z}\}$, $\{S_{4x}, S_{4y}, S_{4z}\}$. We say that such a polynomial is multi-homogeneous and write respectively $d_2(P), d_3(P)$, $d_4(P)$ for the partial degrees with respect to the three sets of variables. We denote by $Bas(d_2, d_3, d_4)$ the set of all the monomials in the basic invariants of partial degrees d_2, d_3, d_4.

The general structure of the algorithm for computing auxiliary invariants reads as follows. As already alluded at, the algorithm is by induction (with respect to the degrees of generators). The algorithm constructs for each multi-degree (d_2, d_3, d_4) a complete set $Aux(d_2, d_3, d_4)$ of auxiliary invariants.

(1) Initialization of the algorithm: compute, inductively, for all multidegrees (d_2, d_3, d_4) the corresponding set of multi-homogeneous monomials in the basic, algebraically independent, invariants: $Bas(d_2, d_3, d_4)$.
Set $Aux(d_2, d_3, d_4) = \{1\}$ for $(d_2, d_3, d_4) = (0, 0, 0)$, and $Aux(d_2, d_3, d_4) = \emptyset$ in all other cases.
(2) For $1 \leq n \leq 12$ ($= \max deg(g_i)$ appearing in the Molien series numerator, see Eq. 18), assume also that the auxiliary invariants of total degree $n - 1$ have been

constructed. Put the lexicographical order on the multi-degrees (d_2, d_3, d_4), *such that* $d_2 + d_3 + d_4 = n$.

(3) For $(0, 1, n-1) \leq (d_2, d_3, d_4) \leq (n-1, 1, 0)$ (to skip some multidegrees, use the fact that according to Theorem 1 there cannot be any auxiliary invariant with only one $d_i \neq 0$ for XY_4 molecule), construct all the invariant monomials in the basic invariants and the auxiliary invariants that can be obtained as the product of an element of $Bas(d_2', d_3', d_4')$ with an element of $Aux(d_2'', d_3'', d_4'')$ such that $d_2 = d_2' + d_2'', d_3 = d_3' + d_3'', d_4 = d_4' + d_4''$. Call $Inv(d_2, d_3, d_4)$ this set of monomials.

(4) Using the Reynolds operator associated to the nuclear symmetric group $\mathcal{S}_4$,

$$\frac{1}{Card(G)} \sum_{g \in \mathcal{S}_4} g$$

which is a projector from the algebra of polynomials in the $S_{2a}, \ldots, S_{4z}$ to the algebra of $\mathcal{S}_4$-polynomial invariants, construct an ordered set of generators $B(d_2, d_3, d_4) = \{b_1, \ldots, b_k\}$ for the vector space of $\mathcal{S}_4$-polynomial invariants of multi-degree (d_2, d_3, d_4).

(5) Test recursively if the elements of $B(d_2, d_3, d_4)$ belong to the linear span of $Inv(d_2, d_3, d_4)$. If the element b_i belongs to $Inv(d_2, d_3, d_4)$, proceed to b_{i+1} as long as $i < k$. Else, set $Inv(d_2, d_3, d_4) := Inv(d_2, d_3, d_4) \cup \{b_i\}$, $Aux(d_2, d_3, d_4) := Aux(d_2, d_3, d_4) \cup \{b_i\}$ and proceed to b_{i+1} as long as $i < k$. When all the b_i have been considered, a complete family of auxiliary invariants of multi degree (d_2, d_3, d_4) has been obtained, $Aux(d_2, d_3, d_4)$.

(6) Proceed to the next multidegree (d_2, d_3, d_4), in the lexicographical order. If $(d_2, d_3, d_4) \leq (n-1, 1, 0)$ goto point (3). If $(d_2, d_3, d_4) = (n, 0, 0)$ and $n < 12$ increment n and goto point (2). Finally, if $(d_2, d_3, d_4) = (12, 0, 0)$ the Molien series Eq. 18 allows one to conclude that all the auxiliary invariants have been found. Terminate the process.

The complete list of the 143 auxiliary invariants is given in Appendix.

4 Conclusion

We have determined for the first time an integrity basis of the $O(3) \times \mathcal{S}_4$-invariant subalgebra of the polynomial algebra over the translation-free Cartesians coordinates. It is composed of 9 algebraically independent, basic invariants and 575 auxiliary invariants (S_5, S_5^2, S_5^3 the 143 invariants given in Appendix, and their products with S_5, S_5^2, S_5^3). These polynomials can be used to express (the smooth part of) symmetry-adapted, global PES for XY_4-type of molecules.

In this work, we have taken advantage of specific features of XY_4 molecules. For example, we have used the fact that the action of $\mathcal{S}_4$ on the degree 2 $O(3)$-invariant polynomials in the cartesian coordinates splits into subrepresentations on which the action of $\mathcal{S}_4$ is generated by pseudo-reflections. If this had not been the case, the multidegree indices in point 3 of the algorithm of Sect. 4 would have had to run over all possibilities, including the case where all the partial degrees but one are zero. However, the strategy followed to derive the invariants is general, since for all finite group, G, the $O(3)\times G$-invariant polynomials form a Cohen-Macaulay algebra by the theorems of Hochster and Eagon and Hochster and Roberts [7,8].

Molien series derivation is not compulsory in our approach. It is used essentially to know when to stop the algorithm. However, there are other ways to determine upper bounds on the maximal degree of the auxiliary invariants and the maximal number of the latter.

In fact, our approach makes available for the study of global PES. the recent tools of ring and invariant theory such as Cohen-Macaulay-type properties and the related effective computational tools of modern commutative algebra [6], which go far beyond the classical Molien series approach in quantum chemistry.

In the near future, we will extend our approach to the case of more general type of symmetry-adapted polynomials, in order to deal with dipole moment surfaces in a similar fashion.

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Appendix

Auxiliary invariants for XY_4

Degree 2; card = 1

$$A_1^{0,1,1} := \frac{S_{3x}S_{4x} + S_{3y}S_{4y} + S_{3z}S_{4z}}{\sqrt{3}}$$

Degree 3; card = 5

$$A_1^{0,1,2} := \frac{S_{3z}S_{4x}S_{4y} + S_{3y}S_{4x}S_{4z} + S_{3x}S_{4y}S_{4z}}{\sqrt{3}}$$

$$A_1^{0,2,1} := \frac{S_{3y}S_{3z}S_{4x} + S_{3x}S_{3z}S_{4y} + S_{3x}S_{3y}S_{4z}}{\sqrt{3}}$$

$$A_1^{1,0,2} := \frac{1}{6}\left(3S_{2b}\left(-S_{4x}^2 + S_{4y}^2\right) + \sqrt{3}S_{2a}\left(S_{4x}^2 + S_{4y}^2 - 2S_{4z}^2\right)\right)$$

$$A_1^{1,1,1} := \frac{1}{6}\left(3S_{2b}(-S_{3x}S_{4x} + S_{3y}S_{4y}) + \sqrt{3}S_{2a}(S_{3x}S_{4x} + S_{3y}S_{4y} - 2S_{3z}S_{4z})\right)$$

$$A_1^{1,2,0} := \frac{1}{6}\left(3S_{2b}\left(-S_{3x}^2 + S_{3y}^2\right) + \sqrt{3}S_{2a}\left(S_{3x}^2 + S_{3y}^2 - 2S_{3z}^2\right)\right)$$

Degree 4; card = 9

$$A_1^{0,1,3} := \frac{S_{3x}S_{4x}^3 + S_{3y}S_{4y}^3 + S_{3z}S_{4z}^3}{\sqrt{3}}$$

$$A_1^{0,2,2} := \frac{S_{3x}^2S_{4x}^2 + S_{3y}^2S_{4y}^2 + S_{3z}^2S_{4z}^2}{\sqrt{3}}$$

$$A_2^{0,2,2} := \frac{S_{3y}S_{3z}S_{4y}S_{4z} + S_{3x}S_{4x}(S_{3y}S_{4y} + S_{3z}S_{4z})}{\sqrt{3}}$$

$$A_1^{0,3,1} := \frac{S_{3x}^3 S_{4x} + S_{3y}^3 S_{4y} + S_{3z}^3 S_{4z}}{\sqrt{3}}$$

$$A_1^{1,1,2} := \frac{1}{6}\big(3S_{2b}(S_{3y}S_{4x} - S_{3x}S_{4y})S_{4z} + \sqrt{3}S_{2a}\big(-2S_{3z}S_{4x}S_{4y} + S_{3y}S_{4x}S_{4z} + S_{3x}S_{4y}S_{4z}\big)\big)$$

$$A_1^{1,2,1} := \frac{1}{6}\big(3S_{2b}S_{3z}(S_{3y}S_{4x} - S_{3x}S_{4y}) - \sqrt{3}S_{2a}\big(S_{3y}S_{3z}S_{4x} + S_{3x}S_{3z}S_{4y} - 2S_{3x}S_{3y}S_{4z}\big)\big)$$

$$A_1^{2,0,2} := \frac{1}{6\sqrt{5}}\big(6S_{2a}S_{2b}\big(S_{4x}^2 - S_{4y}^2\big) + 3\sqrt{3}S_{2a}^2\big(S_{4x}^2 + S_{4y}^2\big) + \sqrt{3}S_{2b}^2\big(S_{4x}^2 + S_{4y}^2 + 4S_{4z}^2\big)\big)$$

$$A_1^{2,1,1} := \frac{1}{6\sqrt{5}}\big(6S_{2a}S_{2b}(S_{3x}S_{4x} - S_{3y}S_{4y}) + 3\sqrt{3}S_{2a}^2(S_{3x}S_{4x} + S_{3y}S_{4y}) + \sqrt{3}S_{2b}^2(S_{3x}S_{4x} + S_{3y}S_{4y} + 4S_{3z}S_{4z})\big)$$

$$A_1^{2,2,0} := \frac{1}{6\sqrt{5}}\big(6S_{2a}S_{2b}\big(S_{3x}^2 - S_{3y}^2\big) + 3\sqrt{3}S_{2a}^2\big(S_{3x}^2 + S_{3y}^2\big) + \sqrt{3}S_{2b}^2\big(S_{3x}^2 + S_{3y}^2 + 4S_{3z}^2\big)\big)$$

Degree 5; Card = 12

$$A_1^{0,2,3} := \frac{1}{\sqrt{6}}\big(S_{3y}S_{3z}S_{4x}\big(S_{4y}^2 + S_{4z}^2\big) + S_{3x}\big(S_{3y}\big(S_{4x}^2 + S_{4y}^2\big)S_{4z} + S_{3z}S_{4y}\big(S_{4x}^2 + S_{4z}^2\big)\big)\big)$$

$$A_1^{0,3,2} := \frac{S_{3z}^3 S_{4x}S_{4y} + S_{3y}^3 S_{4x}S_{4z} + S_{3x}^3 S_{4y}S_{4z}}{\sqrt{3}}$$

$$A_1^{1,0,4} := \frac{1}{6}\big(3S_{2b}\big(-S_{4x}^2 + S_{4y}^2\big)S_{4z}^2 - \sqrt{3}S_{2a}\big(S_{4y}^2S_{4z}^2 + S_{4x}^2\big(-2S_{4y}^2 + S_{4z}^2\big)\big)\big)$$

$$A_1^{1,1,3} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3y}S_{4y}\big(S_{4x}^2 - S_{4z}^2\big) + S_{3x}S_{4x}\big(S_{4y}^2 - S_{4z}^2\big)\big) + \sqrt{3}S_{2b}\big(2S_{3z}\big(-S_{4x}^2 + S_{4y}^2\big)S_{4z} + S_{3x}S_{4x}\big(S_{4y}^2 - S_{4z}^2\big) + S_{3y}S_{4y}\big(-S_{4x}^2 + S_{4z}^2\big)\big)\big)$$

$$A_2^{1,1,3} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3y}S_{4x}^2S_{4y} + S_{3x}S_{4x}S_{4y}^2 - S_{3z}\big(S_{4x}^2 + S_{4y}^2\big)S_{4z}\big) + \sqrt{3}S_{2b}\big(S_{3z}\big(-S_{4x}^2 + S_{4y}^2\big)S_{4z} + S_{3y}S_{4y}\big(S_{4x}^2 + 2S_{4z}^2\big) - S_{3x}S_{4x}\big(S_{4y}^2 + 2S_{4z}^2\big)\big)\big)$$

$$A_1^{1,2,2} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3y}^2\big(S_{4x}^2 - S_{4z}^2\big) + S_{3x}^2\big(S_{4y}^2 - S_{4z}^2\big)\big)$$
$$+\sqrt{3}S_{2b}\big(-2S_{3z}^2\big(S_{4x}^2 - S_{4y}^2\big) + S_{3x}^2\big(S_{4y}^2 - S_{4z}^2\big) + S_{3y}^2\big(-S_{4x}^2 + S_{4z}^2\big)\big)\big)$$
$$A_2^{1,2,2} := \frac{1}{6}\big(3S_{2b}S_{3z}(-S_{3x}S_{4x} + S_{3y}S_{4y})S_{4z} - \sqrt{3}S_{2a}\big(-2S_{3x}S_{3y}S_{4x}S_{4y}$$
$$+S_{3x}S_{3z}S_{4x}S_{4z} + S_{3y}S_{3z}S_{4y}S_{4z}\big)\big)$$
$$A_1^{1,3,1} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3x}S_{3y}^2S_{4x} - S_{3y}^2S_{3z}S_{4z} + S_{3x}^2(S_{3y}S_{4y} - S_{3z}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}\big(-S_{3x}\big(S_{3y}^2 + 2S_{3z}^2\big)S_{4x} + S_{3y}S_{3z}(2S_{3z}S_{4y} + S_{3y}S_{4z})$$
$$+S_{3x}^2(S_{3y}S_{4y} - S_{3z}S_{4z})\big)\big)$$
$$A_2^{1,3,1} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3x}\big(S_{3y}^2 - S_{3z}^2\big)S_{4x} + S_{3x}^2S_{3y}S_{4y} - S_{3y}S_{3z}^2S_{4y}\big)$$
$$+\sqrt{3}S_{2b}\big(S_{3x}\big(S_{3y}^2 - S_{3z}^2\big)S_{4x} + S_{3y}S_{3z}(S_{3z}S_{4y} + 2S_{3y}S_{4z})$$
$$-S_{3x}^2(S_{3y}S_{4y} + 2S_{3z}S_{4z})\big)\big)$$
$$A_1^{1,4,0} := \frac{1}{6}\big(3S_{2b}\big(-S_{3x}^2 + S_{3y}^2\big)S_{3z}^2 - \sqrt{3}S_{2a}\big(S_{3y}^2S_{3z}^2 + S_{3x}^2\big(-2S_{3y}^2 + S_{3z}^2\big)\big)\big)$$
$$A_1^{2,1,2} := \frac{1}{6\sqrt{5}}\big(6S_{2a}S_{2b}(-S_{3y}S_{4x} + S_{3x}S_{4y})S_{4z} + 3\sqrt{3}S_{2a}^2(S_{3y}S_{4x} + S_{3x}S_{4y})S_{4z}$$
$$+\sqrt{3}S_{2b}^2(4S_{3z}S_{4x}S_{4y} + S_{3y}S_{4x}S_{4z} + S_{3x}S_{4y}S_{4z})\big)$$
$$A_1^{2,2,1} := \frac{1}{6\sqrt{11}}\big(6S_{2a}S_{2b}S_{3z}(-S_{3y}S_{4x} + S_{3x}S_{4y}) + 3\sqrt{3}S_{2a}^2\big(S_{3y}S_{3z}S_{4x}$$
$$+S_{3x}S_{3z}S_{4y} + 2S_{3x}S_{3y}S_{4z}\big) + \sqrt{3}S_{2b}^2\big(5S_{3y}S_{3z}S_{4x} + 5S_{3x}S_{3z}S_{4y}$$
$$+2S_{3x}S_{3y}S_{4z}\big)\big)$$

Degree 6; Card = 18

$$A_1^{0,2,4} := \frac{S_{3x}^2S_{4x}^4 + S_{3y}^2S_{4y}^4 + S_{3z}^2S_{4z}^4}{\sqrt{3}}$$
$$A_1^{0,3,3} := \frac{S_{3x}^3S_{4x}^3 + S_{3y}^3S_{4y}^3 + S_{3z}^3S_{4z}^3}{\sqrt{3}}$$
$$A_2^{0,3,3} := \frac{1}{\sqrt{6}}\big(S_{3x}^2S_{4x}^2(S_{3y}S_{4y} + S_{3z}S_{4z}) + S_{3y}S_{3z}S_{4y}S_{4z}(S_{3y}S_{4y} + S_{3z}S_{4z})$$
$$+S_{3x}S_{4x}\big(S_{3y}^2S_{4y}^2 + S_{3z}^2S_{4z}^2\big)\big)$$
$$A_1^{0,4,2} := \frac{S_{3x}^4S_{4x}^2 + S_{3y}^4S_{4y}^2 + S_{3z}^4S_{4z}^2}{\sqrt{3}}$$

$$\begin{aligned}
A_1^{1,1,4} := \frac{1}{6\sqrt{2}}&\big(\sqrt{3}S_{2b}\big(-2S_{3z}S_{4x}^3S_{4y}+2S_{3z}S_{4x}S_{4y}^3-S_{3y}S_{4x}^3S_{4z}+S_{3x}S_{4y}^3S_{4z}\\
&+S_{3y}S_{4x}S_{4z}^3-S_{3x}S_{4y}S_{4z}^3\big)+3S_{2a}S_{4z}\big(S_{3x}S_{4y}\big(S_{4y}^2-S_{4z}^2\big)\\
&+S_{3y}\big(S_{4x}^3-S_{4x}S_{4z}^2\big)\big)\big)\\
A_1^{1,2,3} := \frac{1}{6\sqrt{2}}&\big(-3S_{2a}\big(S_{3y}S_{3z}S_{4x}S_{4y}^2+S_{3x}\big(S_{3z}S_{4x}^2S_{4y}-S_{3y}\big(S_{4x}^2+S_{4y}^2\big)S_{4z}\big)\big)\\
&+\sqrt{3}S_{2b}\big(S_{3y}S_{3z}S_{4x}\big(S_{4y}^2+2S_{4z}^2\big)-S_{3x}\big(S_{3y}\big(-S_{4x}^2+S_{4y}^2\big)S_{4z}\\
&+S_{3z}S_{4y}\big(S_{4x}^2+2S_{4z}^2\big)\big)\big)\big)\\
A_2^{1,2,3} := \frac{1}{6\sqrt{2}}&\big(3S_{2a}S_{4z}\big(-S_{3y}S_{3z}S_{4x}S_{4z}+S_{3x}\big(S_{3y}\big(S_{4x}^2+S_{4y}^2\big)-S_{3z}S_{4y}S_{4z}\big)\big)\\
&+\sqrt{3}S_{2b}\big(S_{3y}S_{3z}S_{4x}\big(2S_{4y}^2+S_{4z}^2\big)-S_{3x}\big(S_{3y}\big(S_{4x}^2-S_{4y}^2\big)S_{4z}\\
&+S_{3z}S_{4y}\big(2S_{4x}^2+S_{4z}^2\big)\big)\big)\big)\\
A_1^{1,3,2} := \frac{1}{6\sqrt{2}}&\big(-3S_{2a}\big(S_{3y}^2S_{3z}S_{4x}S_{4y}-S_{3x}S_{3y}^2S_{4y}S_{4z}+S_{3x}^2S_{4x}(S_{3z}S_{4y}-S_{3y}S_{4z})\big)\\
&+\sqrt{3}S_{2b}\big(-S_{3x}\big(S_{3y}^2+2S_{3z}^2\big)S_{4y}S_{4z}+S_{3y}S_{3z}S_{4x}(S_{3y}S_{4y}+2S_{3z}S_{4z})\\
&+S_{3x}^2(-S_{3z}S_{4x}S_{4y}+S_{3y}S_{4x}S_{4z})\big)\big)\\
A_2^{1,3,2} := \frac{1}{6\sqrt{2}}&\big(3S_{2a}\big(S_{3x}^2S_{3y}S_{4x}-S_{3y}S_{3z}^2S_{4x}+S_{3x}\big(S_{3y}^2-S_{3z}^2\big)S_{4y}\big)S_{4z}\\
&+\sqrt{3}S_{2b}\big(S_{3x}\big(S_{3y}^2-S_{3z}^2\big)S_{4y}S_{4z}-S_{3x}^2S_{4x}(2S_{3z}S_{4y}+S_{3y}S_{4z})\\
&+S_{3y}S_{3z}S_{4x}(2S_{3y}S_{4y}+S_{3z}S_{4z})\big)\big)\\
A_1^{1,4,1} := \frac{1}{6\sqrt{2}}&\big(-3S_{2a}\big(S_{3x}^3S_{3z}S_{4y}-S_{3x}^3S_{3y}S_{4z}+S_{3y}^3(S_{3z}S_{4x}-S_{3x}S_{4z})\big)\\
&+\sqrt{3}S_{2b}\big(-S_{3z}\big(S_{3x}^3+2S_{3x}S_{3z}^2\big)S_{4y}+S_{3y}^3(S_{3z}S_{4x}-S_{3x}S_{4z})\\
&+S_{3y}\big(2S_{3z}^3S_{4x}+S_{3x}^3S_{4z}\big)\big)\big)\\
A_1^{2,0,4} := \frac{1}{6\sqrt{5}}&\big(6S_{2a}S_{2b}\big(S_{4x}^4-S_{4y}^4\big)+3\sqrt{3}S_{2a}^2\big(S_{4x}^4+S_{4y}^4\big)\\
&+\sqrt{3}S_{2b}^2\big(S_{4x}^4+S_{4y}^4+4S_{4z}^4\big)\big)\\
A_1^{2,1,3} := \frac{1}{6\sqrt{5}}&\big(6S_{2a}S_{2b}\big(S_{3x}S_{4x}^3-S_{3y}S_{4y}^3\big)+3\sqrt{3}S_{2a}^2\big(S_{3x}S_{4x}^3+S_{3y}S_{4y}^3\big)\\
&+\sqrt{3}S_{2b}^2\big(S_{3x}S_{4x}^3+S_{3y}S_{4y}^3+4S_{3z}S_{4z}^3\big)\big)\\
A_2^{2,1,3} := \frac{1}{6\sqrt{10}}&\big(6S_{2a}S_{2b}\big(S_{3y}S_{4x}^2S_{4y}-S_{3x}S_{4x}S_{4y}^2+S_{3z}\big(S_{4x}^2-S_{4y}^2\big)S_{4z}\big)\\
&+3\sqrt{3}S_{2a}^2\big(S_{3y}S_{4x}^2S_{4y}+S_{3x}S_{4x}S_{4y}^2+S_{3z}\big(S_{4x}^2+S_{4y}^2\big)S_{4z}\big)\\
&+\sqrt{3}S_{2b}^2\big(S_{3z}\big(S_{4x}^2+S_{4y}^2\big)S_{4z}+S_{3y}S_{4y}\big(S_{4x}^2+4S_{4z}^2\big)\\
&+S_{3x}S_{4x}\big(S_{4y}^2+4S_{4z}^2\big)\big)\big)
\end{aligned}$$

$$A_1^{2,2,2} := \frac{1}{6\sqrt{5}}\big(6S_{2a}S_{2b}\big(S_{3x}^2S_{4x}^2 - S_{3y}^2S_{4y}^2\big) + 3\sqrt{3}S_{2a}^2\big(S_{3x}^2S_{4x}^2 + S_{3y}^2S_{4y}^2\big) + \sqrt{3}S_{2b}^2\big(S_{3x}^2S_{4x}^2 + S_{3y}^2S_{4y}^2 + 4S_{3z}^2S_{4z}^2\big)\big)$$

$$A_2^{2,2,2} := \frac{1}{6\sqrt{11}}\big(6S_{2a}S_{2b}S_{3z}(S_{3x}S_{4x} - S_{3y}S_{4y})S_{4z} + 3\sqrt{3}S_{2a}^2(S_{3y}S_{3z}S_{4y}S_{4z} + S_{3x}S_{4x}(2S_{3y}S_{4y} + S_{3z}S_{4z})) + \sqrt{3}S_{2b}^2(5S_{3y}S_{3z}S_{4y}S_{4z} + S_{3x}S_{4x}(2S_{3y}S_{4y} + 5S_{3z}S_{4z}))\big)$$

$$A_1^{2,3,1} := \frac{1}{6\sqrt{5}}\big(6S_{2a}S_{2b}\big(S_{3x}^3S_{4x} - S_{3y}^3S_{4y}\big) + 3\sqrt{3}S_{2a}^2\big(S_{3x}^3S_{4x} + S_{3y}^3S_{4y}\big) + \sqrt{3}S_{2b}^2\big(S_{3x}^3S_{4x} + S_{3y}^3S_{4y} + 4S_{3z}^3S_{4z}\big)\big)$$

$$A_2^{2,3,1} := \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3x}S_{3y}^2S_{4x} - S_{3y}^2S_{3z}S_{4z} + S_{3x}^2(S_{3y}S_{4y} + S_{3z}S_{4z})\big) + 3\sqrt{3}S_{2a}^2\big(S_{3x}S_{3y}^2S_{4x} + S_{3y}^2S_{3z}S_{4z} + S_{3x}^2(S_{3y}S_{4y} + S_{3z}S_{4z})\big) + \sqrt{3}S_{2b}^2\big(S_{3x}\big(S_{3y}^2 + 4S_{3z}^2\big)S_{4x} + S_{3y}S_{3z}(4S_{3z}S_{4y} + S_{3y}S_{4z}) + S_{3x}^2(S_{3y}S_{4y} + S_{3z}S_{4z})\big)\big)$$

$$A_1^{2,4,0} := \frac{1}{6\sqrt{5}}\big(6S_{2a}S_{2b}\big(S_{3x}^4 - S_{3y}^4\big) + 3\sqrt{3}S_{2a}^2\big(S_{3x}^4 + S_{3y}^4\big) + \sqrt{3}S_{2b}^2\big(S_{3x}^4 + S_{3y}^4 + 4S_{3z}^4\big)\big)$$

Degree 7; Card = 21

$$A_1^{0,3,4} := \frac{1}{\sqrt{6}}\big(S_{3x}^2S_{4x}^3(S_{3z}S_{4y} + S_{3y}S_{4z}) + S_{3y}S_{3z}S_{4x}\big(S_{3y}S_{4y}^3 + S_{3z}S_{4z}^3\big) + S_{3x}\big(S_{3y}^2S_{4y}^3S_{4z} + S_{3z}^2S_{4y}S_{4z}^3\big)\big)$$

$$A_1^{0,4,3} := \frac{1}{\sqrt{6}}\big(S_{3x}^3S_{4x}^2(S_{3z}S_{4y} + S_{3y}S_{4z}) + S_{3x}S_{4y}S_{4z}\big(S_{3y}^3S_{4y} + S_{3z}^3S_{4z}\big) + S_{3y}S_{3z}S_{4x}\big(S_{3y}^2S_{4y}^2 + S_{3z}^2S_{4z}^2\big)\big)$$

$$A_1^{1,1,5} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3y}S_{4y}^3\big(S_{4x}^2 - S_{4z}^2\big) + S_{3x}S_{4x}^3\big(S_{4y}^2 - S_{4z}^2\big)\big) + \sqrt{3}S_{2b}\big(2S_{3z}\big(-S_{4x}^2 + S_{4y}^2\big)S_{4z}^3 + S_{3x}S_{4x}^3\big(S_{4y}^2 - S_{4z}^2\big) + S_{3y}S_{4y}^3\big(-S_{4x}^2 + S_{4z}^2\big)\big)\big)$$

$$A_1^{1,2,4} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3y}^2S_{4y}^2\big(S_{4x}^2 - S_{4z}^2\big) + S_{3x}^2S_{4x}^2\big(S_{4y}^2 - S_{4z}^2\big)\big) + \sqrt{3}S_{2b}\big(2S_{3z}^2\big(-S_{4x}^2 + S_{4y}^2\big)S_{4z}^2 + S_{3x}^2S_{4x}^2\big(S_{4y}^2 - S_{4z}^2\big) + S_{3y}^2S_{4y}^2\big(-S_{4x}^2 + S_{4z}^2\big)\big)\big)$$

$$A_2^{1,2,4} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(-S_{3y}S_{3z}S_{4y}^3S_{4z}+S_{3x}\big(S_{3y}S_{4x}S_{4y}\big(S_{4x}^2+S_{4y}^2\big)-S_{3z}S_{4x}^3S_{4z}\big)\big)$$
$$+\sqrt{3}S_{2b}\big(S_{3y}S_{3z}S_{4y}S_{4z}\big(S_{4y}^2+2S_{4z}^2\big)+S_{3x}S_{4x}\big(S_{3y}S_{4y}\big(S_{4x}^2-S_{4y}^2\big)$$
$$-S_{3z}S_{4z}\big(S_{4x}^2+2S_{4z}^2\big)\big)\big)\big)$$

$$A_1^{1,3,3} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3y}^3S_{4y}\big(S_{4x}^2-S_{4z}^2\big)+S_{3x}^3S_{4x}\big(S_{4y}^2-S_{4z}^2\big)\big)$$
$$+\sqrt{3}S_{2b}\big(2S_{3z}^3\big(-S_{4x}^2+S_{4y}^2\big)S_{4z}+S_{3x}^3S_{4x}\big(S_{4y}^2-S_{4z}^2\big)$$
$$+S_{3y}^3S_{4y}\big(-S_{4x}^2+S_{4z}^2\big)\big)\big)$$

$$A_2^{1,3,3} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3x}S_{3y}^2S_{4x}S_{4y}^2-S_{3y}^2S_{3z}S_{4y}^2S_{4z}+S_{3x}^2S_{4x}^2(S_{3y}S_{4y}-S_{3z}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}\big(S_{3x}^2S_{4x}^2(S_{3y}S_{4y}-S_{3z}S_{4z})+S_{3y}S_{3z}S_{4y}S_{4z}(S_{3y}S_{4y}+2S_{3z}S_{4z})$$
$$-S_{3x}S_{4x}\big(S_{3y}^2S_{4y}^2+2S_{3z}^2S_{4z}^2\big)\big)\big)$$

$$A_3^{1,3,3} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3x}S_{3y}^2S_{4x}^3-S_{3y}^2S_{3z}S_{4z}^3+S_{3x}^2\big(S_{3y}S_{4y}^3-S_{3z}S_{4z}^3\big)\big)$$
$$+\sqrt{3}S_{2b}\big(-S_{3x}\big(S_{3y}^2+2S_{3z}^2\big)S_{4x}^3+S_{3y}S_{3z}\big(2S_{3z}S_{4y}^3+S_{3y}S_{4z}^3\big)$$
$$+S_{3x}^2\big(S_{3y}S_{4y}^3-S_{3z}S_{4z}^3\big)\big)\big)$$

$$A_4^{1,3,3} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3x}\big(S_{3y}^2-S_{3z}^2\big)S_{4x}^3+S_{3x}^2S_{3y}S_{4y}^3-S_{3y}S_{3z}^2S_{4y}^3\big)$$
$$+\sqrt{3}S_{2b}\big(S_{3x}\big(S_{3y}^2-S_{3z}^2\big)S_{4x}^3+S_{3y}S_{3z}\big(S_{3z}S_{4y}^3+2S_{3y}S_{4z}^3\big)$$
$$-S_{3x}^2\big(S_{3y}S_{4y}^3+2S_{3z}S_{4z}^3\big)\big)\big)$$

$$A_1^{1,4,2} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3y}^4\big(S_{4x}^2-S_{4z}^2\big)+S_{3x}^4\big(S_{4y}^2-S_{4z}^2\big)\big)$$
$$+\sqrt{3}S_{2b}\big(-2S_{3z}^4\big(S_{4x}^2-S_{4y}^2\big)+S_{3x}^4\big(S_{4y}^2-S_{4z}^2\big)+S_{3y}^4\big(-S_{4x}^2+S_{4z}^2\big)\big)\big)$$

$$A_2^{1,4,2} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3x}S_{3y}^3S_{4x}S_{4y}-S_{3y}^3S_{3z}S_{4y}S_{4z}+S_{3x}^3S_{4x}(S_{3y}S_{4y}-S_{3z}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}\big(S_{3y}S_{3z}\big(S_{3y}^2+2S_{3z}^2\big)S_{4y}S_{4z}+S_{3x}^3S_{4x}(S_{3y}S_{4y}-S_{3z}S_{4z})$$
$$-S_{3x}S_{4x}\big(S_{3y}^3S_{4y}+2S_{3z}^3S_{4z}\big)\big)\big)$$

$$A_1^{1,5,1} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3x}S_{3y}^4S_{4x}-S_{3y}^4S_{3z}S_{4z}+S_{3x}^4(S_{3y}S_{4y}-S_{3z}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}\big(-S_{3x}\big(S_{3y}^4+2S_{3z}^4\big)S_{4x}+2S_{3y}S_{3z}^4S_{4y}+S_{3y}^4S_{3z}S_{4z}$$
$$+S_{3x}^4(S_{3y}S_{4y}-S_{3z}S_{4z})\big)\big)$$

$$A_1^{2,1,4} := \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(S_{3z}S_{4x}^3S_{4y}-S_{3z}S_{4x}S_{4y}^3+S_{3y}S_{4x}^3S_{4z}-S_{3x}S_{4y}^3S_{4z}\big)$$
$$+3\sqrt{3}S_{2a}^2\big(S_{3z}S_{4x}S_{4y}\big(S_{4x}^2+S_{4y}^2\big)+\big(S_{3y}S_{4x}^3+S_{3x}S_{4y}^3\big)S_{4z}\big)$$

$$+\sqrt{3}S_{2b}^2\big(S_{3z}S_{4x}S_{4y}\big(S_{4x}^2+S_{4y}^2\big)$$
$$+S_{4z}\big(S_{3y}S_{4x}^3+S_{3x}S_{4y}^3+4S_{3y}S_{4x}S_{4z}^2+4S_{3x}S_{4y}S_{4z}^2\big)\big)\big)$$

$$A_1^{2,2,3} := \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3y}S_{3z}S_{4x}S_{4y}^2+S_{3x}\big(S_{3z}S_{4x}^2S_{4y}$$
$$+S_{3y}\big(S_{4x}^2-S_{4y}^2\big)S_{4z}\big)\big)+3\sqrt{3}S_{2a}^2\big(S_{3y}S_{3z}S_{4x}S_{4y}^2$$
$$+S_{3x}\big(S_{3z}S_{4x}^2S_{4y}+S_{3y}\big(S_{4x}^2+S_{4y}^2\big)S_{4z}\big)\big)$$
$$+\sqrt{3}S_{2b}^2\big(S_{3y}S_{3z}S_{4x}\big(S_{4y}^2+4S_{4z}^2\big)$$
$$+S_{3x}\big(S_{3y}\big(S_{4x}^2+S_{4y}^2\big)S_{4z}+S_{3z}S_{4y}\big(S_{4x}^2+4S_{4z}^2\big)\big)\big)\big)$$

$$A_2^{2,2,3} := \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}S_{4z}\big(-S_{3y}S_{3z}S_{4x}S_{4z}+S_{3x}\big(S_{3y}\big(-S_{4x}^2+S_{4y}^2\big)$$
$$+S_{3z}S_{4y}S_{4z}\big)\big)+3\sqrt{3}S_{2a}^2S_{4z}\big(S_{3y}S_{3z}S_{4x}S_{4z}+S_{3x}\big(S_{3y}\big(S_{4x}^2+S_{4y}^2\big)$$
$$+S_{3z}S_{4y}S_{4z}\big)\big)+\sqrt{3}S_{2b}^2\big(S_{3y}S_{3z}S_{4x}\big(4S_{4y}^2+S_{4z}^2\big)$$
$$+S_{3x}\big(S_{3y}\big(S_{4x}^2+S_{4y}^2\big)S_{4z}+S_{3z}S_{4y}\big(4S_{4x}^2+S_{4z}^2\big)\big)\big)\big)$$

$$A_1^{2,3,2} := \frac{1}{6\sqrt{5}}\big(6S_{2a}S_{2b}\big(-S_{3y}^3S_{4x}+S_{3x}^3S_{4y}\big)S_{4z}+3\sqrt{3}S_{2a}^2\big(S_{3y}^3S_{4x}+S_{3x}^3S_{4y}\big)S_{4z}$$
$$+\sqrt{3}S_{2b}^2\big(4S_{3z}^3S_{4x}S_{4y}+S_{3y}^3S_{4x}S_{4z}+S_{3x}^3S_{4y}S_{4z}\big)\big)$$

$$A_2^{2,3,2} := \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3y}^2S_{3z}S_{4x}S_{4y}-S_{3x}S_{3y}^2S_{4y}S_{4z}+S_{3x}^2S_{4x}(S_{3z}S_{4y}+S_{3y}S_{4z})\big)$$
$$+3\sqrt{3}S_{2a}^2\big(S_{3y}^2S_{3z}S_{4x}S_{4y}+S_{3x}S_{3y}^2S_{4y}S_{4z}+S_{3x}^2S_{4x}(S_{3z}S_{4y}+S_{3y}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}^2\big(S_{3x}\big(S_{3y}^2+4S_{3z}^2\big)S_{4y}S_{4z}+S_{3x}^2S_{4x}$$
$$\times\big(S_{3z}S_{4y}+S_{3y}S_{4z}\big)+S_{3y}S_{3z}S_{4x}(S_{3y}S_{4y}+4S_{3z}S_{4z})\big)\big)$$

$$A_1^{2,4,1} := \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(S_{3x}^3S_{3z}S_{4y}+S_{3x}^3S_{3y}S_{4z}-S_{3y}^3(S_{3z}S_{4x}+S_{3x}S_{4z})\big)$$
$$+3\sqrt{3}S_{2a}^2\big(S_{3x}^3S_{3z}S_{4y}+S_{3x}^3S_{3y}S_{4z}+S_{3y}^3(S_{3z}S_{4x}+S_{3x}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}^2\big(S_{3x}S_{3z}\big(S_{3x}^2+4S_{3z}^2\big)S_{4y}+S_{3y}^3(S_{3z}S_{4x}+S_{3x}S_{4z})$$
$$+S_{3y}\big(4S_{3z}^3S_{4x}+S_{3x}^3S_{4z}\big)\big)\big)$$

$$A_1^{3,1,3} := \frac{1}{12\sqrt{14}}\big(9S_{2a}^3\big(S_{3y}S_{4y}\big(S_{4x}^2-S_{4z}^2\big)+S_{3x}S_{4x}\big(S_{4y}^2-S_{4z}^2\big)\big)$$
$$+9S_{2a}S_{2b}^2\big(S_{3y}S_{4y}\big(S_{4x}^2-S_{4z}^2\big)+S_{3x}S_{4x}\big(S_{4y}^2-S_{4z}^2\big)\big)$$
$$+\sqrt{3}S_{2b}^3\big(8S_{3z}\big(-S_{4x}^2+S_{4y}^2\big)S_{4z}+S_{3x}S_{4x}\big(S_{4y}^2-S_{4z}^2\big)$$
$$+S_{3y}S_{4y}\big(-S_{4x}^2+S_{4z}^2\big)\big)-9\sqrt{3}S_{2a}^2S_{2b}\big(S_{3y}S_{4y}\big(S_{4x}^2-S_{4z}^2\big)$$
$$+S_{3x}S_{4x}\big(-S_{4y}^2+S_{4z}^2\big)\big)\big)$$

$$A_1^{3,2,2} := \frac{1}{12\sqrt{14}}\big(9S_{2a}^3\big(S_{3y}^2\big(S_{4x}^2-S_{4z}^2\big)+S_{3x}^2\big(S_{4y}^2-S_{4z}^2\big)\big)$$
$$+9S_{2a}S_{2b}^2\big(S_{3y}^2\big(S_{4x}^2-S_{4z}^2\big)+S_{3x}^2\big(S_{4y}^2-S_{4z}^2\big)\big)$$

$$+\sqrt{3}S_{2b}^3\big(-8S_{3z}^2\big(S_{4x}^2-S_{4y}^2\big)+S_{3x}^2\big(S_{4y}^2-S_{4z}^2\big)+S_{3y}^2\big(-S_{4x}^2+S_{4z}^2\big)\big)$$
$$-9\sqrt{3}S_{2a}^2S_{2b}\big(S_{3y}^2\big(S_{4x}^2-S_{4z}^2\big)+S_{3x}^2\big(-S_{4y}^2+S_{4z}^2\big)\big)\big)$$

$$A_1^{3,3,1} := \frac{1}{12\sqrt{14}}\big(9S_{2a}^3\big(S_{3x}S_{3y}^2S_{4x}-S_{3y}^2S_{3z}S_{4z}+S_{3x}^2(S_{3y}S_{4y}-S_{3z}S_{4z})\big)$$
$$+9S_{2a}S_{2b}^2\big(S_{3x}S_{3y}^2S_{4x}-S_{3y}^2S_{3z}S_{4z}+S_{3x}^2(S_{3y}S_{4y}-S_{3z}S_{4z})\big)$$
$$+9\sqrt{3}S_{2a}^2S_{2b}\big(-S_{3x}S_{3y}^2S_{4x}+S_{3y}^2S_{3z}S_{4z}+S_{3x}^2(S_{3y}S_{4y}-S_{3z}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}^3\big(-S_{3x}\big(S_{3y}^2+8S_{3z}^2\big)S_{4x}+S_{3y}S_{3z}(8S_{3z}S_{4y}+S_{3y}S_{4z})$$
$$+S_{3x}^2(S_{3y}S_{4y}-S_{3z}S_{4z})\big)\big)$$

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$$A_1^{0,3,5} := \frac{1}{\sqrt{6}}\big(S_{3x}^2S_{4x}^4(S_{3y}S_{4y}+S_{3z}S_{4z})+S_{3y}S_{3z}S_{4y}S_{4z}\big(S_{3y}S_{4y}^3+S_{3z}S_{4z}^3\big)$$
$$+S_{3x}S_{4x}\big(S_{3y}^2S_{4y}^4+S_{3z}^2S_{4z}^4\big)\big)$$

$$A_1^{0,4,4} := \frac{S_{3x}^4S_{4x}^4+S_{3y}^4S_{4y}^4+S_{3z}^4S_{4z}^4}{\sqrt{3}}$$

$$A_2^{0,4,4} := \frac{1}{\sqrt{6}}\big(S_{3y}^3S_{3z}S_{4y}^3S_{4z}+S_{3y}S_{3z}^3S_{4y}S_{4z}^3+S_{3x}^3S_{4x}^3(S_{3y}S_{4y}+S_{3z}S_{4z})$$
$$+S_{3x}S_{4x}\big(S_{3y}^3S_{4y}^3+S_{3z}^3S_{4z}^3\big)\big)$$

$$A_1^{0,5,3} := \frac{1}{\sqrt{6}}\big(S_{3x}^4S_{4x}^2(S_{3y}S_{4y}+S_{3z}S_{4z})+S_{3y}S_{3z}S_{4y}S_{4z}\big(S_{3y}^3S_{4y}+S_{3z}^3S_{4z}\big)$$
$$+S_{3x}S_{4x}\big(S_{3y}^4S_{4y}^2+S_{3z}^4S_{4z}^2\big)\big)$$

$$A_1^{1,2,5} := \frac{1}{6\sqrt{2}}\big(-3S_{2a}\big(S_{3y}S_{3z}S_{4x}S_{4y}^4+S_{3x}\big(S_{3z}S_{4x}^4S_{4y}-S_{3y}\big(S_{4x}^4+S_{4y}^4\big)S_{4z}\big)\big)$$
$$+\sqrt{3}S_{2b}\big(S_{3y}S_{3z}S_{4x}\big(S_{4y}^4+2S_{4z}^4\big)-S_{3x}\big(S_{3y}\big(-S_{4x}^4+S_{4y}^4\big)S_{4z}$$
$$+S_{3z}S_{4y}\big(S_{4x}^4+2S_{4z}^4\big)\big)\big)\big)$$

$$A_1^{1,3,4} := \frac{1}{6\sqrt{2}}\big(3S_{2a}S_{4z}\big(S_{3x}^3S_{4y}\big(S_{4y}^2-S_{4z}^2\big)+S_{3y}^3\big(S_{4x}^3-S_{4x}S_{4z}^2\big)\big)$$
$$+\sqrt{3}S_{2b}\big(S_{3z}^3\big(-2S_{4x}^3S_{4y}+2S_{4x}S_{4y}^3\big)+S_{3x}^3S_{4y}S_{4z}\big(S_{4y}^2-S_{4z}^2\big)$$
$$+S_{3y}^3\big(-S_{4x}^3S_{4z}+S_{4x}S_{4z}^3\big)\big)\big)$$

$$A_2^{1,3,4} := \frac{1}{6\sqrt{2}}\big(-3S_{2a}\big(S_{3y}^2S_{3z}S_{4x}S_{4y}^3-S_{3x}S_{3y}^2S_{4y}^3S_{4z}+S_{3x}^2S_{4x}^3(S_{3z}S_{4y}-S_{3y}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}\big(S_{3x}^2S_{4x}^3(-S_{3z}S_{4y}+S_{3y}S_{4z})+S_{3y}S_{3z}S_{4x}\big(S_{3y}S_{4y}^3+2S_{3z}S_{4z}^3\big)$$
$$-S_{3x}\big(S_{3y}^2S_{4y}^3S_{4z}+2S_{3z}^2S_{4y}S_{4z}^3\big)\big)\big)$$

$$A_1^{1,4,3} := \frac{1}{6\sqrt{2}}\big(-3S_{2a}\big(S_{3y}^3S_{3z}S_{4x}S_{4y}^2-S_{3x}S_{3y}^3S_{4y}^2S_{4z}+S_{3x}^3S_{4x}^2(S_{3z}S_{4y}-S_{3y}S_{4z})\big)$$

$$+\sqrt{3}S_{2b}\big(S_{3x}^3S_{4x}^2(-S_{3z}S_{4y}+S_{3y}S_{4z})-S_{3x}S_{4y}S_{4z}\big(S_{3y}^3S_{4y}+2S_{3z}^3S_{4z}\big)$$
$$+S_{3y}S_{3z}S_{4x}\big(S_{3y}^2S_{4y}^2+2S_{3z}^2S_{4z}^2\big)\big)\big)$$

$$A_2^{1,4,3}:=\frac{1}{6\sqrt{2}}\big(3S_{2a}S_{4z}\big(S_{3x}S_{3y}^3S_{4x}^2-S_{3y}^3S_{3z}S_{4x}S_{4z}+S_{3x}^3S_{4y}(S_{3y}S_{4y}-S_{3z}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}\big(-S_{3x}S_{4x}^2\big(2S_{3z}^3S_{4y}+S_{3y}^3S_{4z}\big)+S_{3x}^3S_{4y}S_{4z}(S_{3y}S_{4y}-S_{3z}S_{4z})$$
$$+S_{3y}S_{3z}S_{4x}\big(2S_{3z}^2S_{4y}^2+S_{3y}^2S_{4z}^2\big)\big)\big)$$

$$A_1^{1,5,2}:=\frac{1}{6\sqrt{2}}\big(-3S_{2a}\big(S_{3y}^4S_{3z}S_{4x}S_{4y}-S_{3x}S_{3y}^4S_{4y}S_{4z}+S_{3x}^4S_{4x}(S_{3z}S_{4y}-S_{3y}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}\big(-S_{3x}\big(S_{3y}^4+2S_{3z}^4\big)S_{4y}S_{4z}+S_{3y}S_{3z}S_{4x}\big(S_{3y}^3S_{4y}+2S_{3z}^3S_{4z}\big)$$
$$+S_{3x}^4(-S_{3z}S_{4x}S_{4y}+S_{3y}S_{4x}S_{4z})\big)\big)$$

$$A_1^{2,1,5}:=\frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(S_{3y}S_{4x}^4S_{4y}-S_{3x}S_{4x}S_{4y}^4+S_{3z}\big(S_{4x}^4-S_{4y}^4\big)S_{4z}\big)$$
$$+3\sqrt{3}S_{2a}^2\big(S_{3y}S_{4x}^4S_{4y}+S_{3x}S_{4x}S_{4y}^4+S_{3z}\big(S_{4x}^4+S_{4y}^4\big)S_{4z}\big)$$
$$+\sqrt{3}S_{2b}^2\big(S_{3z}\big(S_{4x}^4+S_{4y}^4\big)S_{4z}+S_{3y}S_{4y}\big(S_{4x}^4+4S_{4z}^4\big)$$
$$+S_{3x}S_{4x}\big(S_{4y}^4+4S_{4z}^4\big)\big)\big)$$

$$A_1^{2,2,4}:=\frac{1}{6\sqrt{5}}\big(6S_{2a}S_{2b}\big(S_{3x}^2S_{4x}^4-S_{3y}^2S_{4y}^4\big)+3\sqrt{3}S_{2a}^2\big(S_{3x}^2S_{4x}^4+S_{3y}^2S_{4y}^4\big)$$
$$+\sqrt{3}S_{2b}^2\big(S_{3x}^2S_{4x}^4+S_{3y}^2S_{4y}^4+4S_{3z}^2S_{4z}^4\big)\big)$$

$$A_2^{2,2,4}:=\frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3y}S_{3z}S_{4y}^3S_{4z}+S_{3x}\big(S_{3y}S_{4x}S_{4y}\big(S_{4x}^2-S_{4y}^2\big)$$
$$+S_{3z}S_{4x}^3S_{4z}\big)\big)+3\sqrt{3}S_{2a}^2\big(S_{3y}S_{3z}S_{4y}^3S_{4z}+S_{3x}\big(S_{3y}S_{4x}S_{4y}\big(S_{4x}^2+S_{4y}^2\big)$$
$$+S_{3z}S_{4x}^3S_{4z}\big)\big)+\sqrt{3}S_{2b}^2\big(S_{3y}S_{3z}S_{4y}S_{4z}\big(S_{4y}^2+4S_{4z}^2\big)$$
$$+S_{3x}S_{4x}\big(S_{3y}S_{4y}\big(S_{4x}^2+S_{4y}^2\big)+S_{3z}S_{4z}\big(S_{4x}^2+4S_{4z}^2\big)\big)\big)\big)$$

$$A_1^{2,3,3}:=\frac{1}{6\sqrt{5}}\big(6S_{2a}S_{2b}\big(S_{3x}^3S_{4x}^3-S_{3y}^3S_{4y}^3\big)+3\sqrt{3}S_{2a}^2\big(S_{3x}^3S_{4x}^3+S_{3y}^3S_{4y}^3\big)$$
$$+\sqrt{3}S_{2b}^2\big(S_{3x}^3S_{4x}^3+S_{3y}^3S_{4y}^3+4S_{3z}^3S_{4z}^3\big)\big)$$

$$A_2^{2,3,3}:=\frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}(S_{3x}S_{4x}-S_{3y}S_{4y})\big(S_{3y}S_{3z}S_{4y}S_{4z}+S_{3x}S_{4x}\big(S_{3y}S_{4y}$$
$$+S_{3z}S_{4z}\big)\big)+3\sqrt{3}S_{2a}^2\big(S_{3x}S_{3y}^2S_{4x}S_{4y}^2+S_{3y}^2S_{3z}S_{4y}^2S_{4z}+S_{3x}^2S_{4x}^2\big(S_{3y}S_{4y}$$
$$+S_{3z}S_{4z}\big)\big)+\sqrt{3}S_{2b}^2\big(S_{3x}^2S_{4x}^2(S_{3y}S_{4y}+S_{3z}S_{4z})$$
$$+S_{3y}S_{3z}S_{4y}S_{4z}(S_{3y}S_{4y}+4S_{3z}S_{4z})+S_{3x}S_{4x}\big(S_{3y}^2S_{4y}^2+4S_{3z}^2S_{4z}^2\big)\big)\big)$$

$$A_3^{2,3,3}:=\frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3x}S_{3y}^2S_{4x}^3-S_{3y}^2S_{3z}S_{4z}^3+S_{3x}^2\big(S_{3y}S_{4y}^3+S_{3z}S_{4z}^3\big)\big)$$
$$+3\sqrt{3}S_{2a}^2\big(S_{3x}S_{3y}^2S_{4x}^3+S_{3y}^2S_{3z}S_{4z}^3+S_{3x}^2\big(S_{3y}S_{4y}^3+S_{3z}S_{4z}^3\big)\big)$$

$$+\sqrt{3}S_{2b}^2\big(S_{3x}\big(S_{3y}^2+4S_{3z}^2\big)S_{4x}^3+S_{3y}S_{3z}\big(4S_{3z}S_{4y}^3+S_{3y}S_{4z}^3\big)$$
$$+S_{3x}^2\big(S_{3y}S_{4y}^3+S_{3z}S_{4z}^3\big)\big)\big)$$

$$A_4^{2,3,3}:=\frac{1}{6\sqrt{10}}\big(-6S_{2a}S_{2b}\big(S_{3x}^2S_{3y}S_{4x}^2S_{4y}+S_{3y}S_{3z}^2S_{4y}S_{4z}^2$$
$$-S_{3x}S_{4x}\big(S_{3y}^2S_{4y}^2+S_{3z}^2S_{4z}^2\big)\big)+3\sqrt{3}S_{2a}^2\big(S_{3x}^2S_{3y}S_{4x}^2S_{4y}+S_{3y}S_{3z}^2S_{4y}S_{4z}^2$$
$$+S_{3x}S_{4x}\big(S_{3y}^2S_{4y}^2+S_{3z}^2S_{4z}^2\big)\big)+\sqrt{3}S_{2b}^2\big(S_{3y}S_{3z}S_{4y}S_{4z}(4S_{3y}S_{4y}+S_{3z}S_{4z})$$
$$+S_{3x}^2S_{4x}^2(S_{3y}S_{4y}+4S_{3z}S_{4z})+S_{3x}S_{4x}\big(S_{3y}^2S_{4y}^2+S_{3z}^2S_{4z}^2\big)\big)\big)$$

$$A_1^{2,4,2}:=\frac{1}{6\sqrt{5}}\big(6S_{2a}S_{2b}\big(S_{3x}^4S_{4x}^2-S_{3y}^4S_{4y}^2\big)$$
$$+3\sqrt{3}S_{2a}^2\big(S_{3x}^4S_{4x}^2+S_{3y}^4S_{4y}^2\big)+\sqrt{3}S_{2b}^2\big(S_{3x}^4S_{4x}^2+S_{3y}^4S_{4y}^2+4S_{3z}^4S_{4z}^2\big)\big)$$

$$A_2^{2,4,2}:=\frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3x}S_{3y}^3S_{4x}S_{4y}-S_{3y}^3S_{3z}S_{4y}S_{4z}+S_{3x}^3S_{4x}\big(S_{3y}S_{4y}$$
$$+S_{3z}S_{4z}\big)\big)+3\sqrt{3}S_{2a}^2\big(S_{3x}S_{3y}^3S_{4x}S_{4y}+S_{3y}^3S_{3z}S_{4y}S_{4z}$$
$$+S_{3x}^3S_{4x}(S_{3y}S_{4y}+S_{3z}S_{4z})\big)+\sqrt{3}S_{2b}^2\big(S_{3y}S_{3z}\big(S_{3y}^2+4S_{3z}^2\big)S_{4y}S_{4z}$$
$$+S_{3x}^3S_{4x}(S_{3y}S_{4y}+S_{3z}S_{4z})+S_{3x}S_{4x}\big(S_{3y}^3S_{4y}+4S_{3z}^3S_{4z}\big)\big)\big)$$

$$A_1^{2,5,1}:=\frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3x}S_{3y}^4S_{4x}-S_{3y}^4S_{3z}S_{4z}+S_{3x}^4(S_{3y}S_{4y}+S_{3z}S_{4z})\big)$$
$$+3\sqrt{3}S_{2a}^2\big(S_{3x}S_{3y}^4S_{4x}+S_{3y}^4S_{3z}S_{4z}+S_{3x}^4(S_{3y}S_{4y}+S_{3z}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}^2\big(S_{3x}\big(S_{3y}^4+4S_{3z}^4\big)S_{4x}+4S_{3y}S_{3z}^4S_{4y}+S_{3y}^4S_{3z}S_{4z}$$
$$+S_{3x}^4(S_{3y}S_{4y}+S_{3z}S_{4z})\big)\big)$$

$$A_1^{3,1,4}:=\frac{1}{12\sqrt{14}}\big(9\sqrt{3}S_{2a}^2S_{2b}S_{4z}\big(-S_{3y}S_{4x}^3+S_{3x}S_{4y}^3+S_{3y}S_{4x}S_{4z}^2-S_{3x}S_{4y}S_{4z}^2\big)$$
$$+\sqrt{3}S_{2b}^3\big(-8S_{3z}S_{4x}^3S_{4y}+8S_{3z}S_{4x}S_{4y}^3-S_{3y}S_{4x}^3S_{4z}+S_{3x}S_{4y}^3S_{4z}$$
$$+S_{3y}S_{4x}S_{4z}^3-S_{3x}S_{4y}S_{4z}^3\big)+9S_{2a}^3S_{4z}\big(S_{3x}S_{4y}\big(S_{4y}^2-S_{4z}^2\big)$$
$$+S_{3y}\big(S_{4x}^3-S_{4x}S_{4z}^2\big)\big)+9S_{2a}S_{2b}^2S_{4z}\big(S_{3x}S_{4y}\big(S_{4y}^2-S_{4z}^2\big)$$
$$+S_{3y}\big(S_{4x}^3-S_{4x}S_{4z}^2\big)\big)\big)$$

$$A_1^{3,2,3}:=\frac{1}{12\sqrt{14}}\big(9\sqrt{3}S_{2a}^2S_{2b}\big(S_{3y}S_{3z}S_{4x}S_{4y}^2-S_{3x}\big(S_{3z}S_{4x}^2S_{4y}$$
$$+S_{3y}\big(-S_{4x}^2+S_{4y}^2\big)S_{4z}\big)\big)-9S_{2a}^3\big(S_{3y}S_{3z}S_{4x}S_{4y}^2+S_{3x}\big(S_{3z}S_{4x}^2S_{4y}$$
$$-S_{3y}\big(S_{4x}^2+S_{4y}^2\big)S_{4z}\big)\big)-9S_{2a}S_{2b}^2\big(S_{3y}S_{3z}S_{4x}S_{4y}^2+S_{3x}\big(S_{3z}S_{4x}^2S_{4y}$$
$$-S_{3y}\big(S_{4x}^2+S_{4y}^2\big)S_{4z}\big)\big)+\sqrt{3}S_{2b}^3\big(S_{3y}S_{3z}S_{4x}\big(S_{4y}^2+8S_{4z}^2\big)$$
$$-S_{3x}\big(S_{3y}\big(-S_{4x}^2+S_{4y}^2\big)S_{4z}+S_{3z}S_{4y}\big(S_{4x}^2+8S_{4z}^2\big)\big)\big)\big)$$

$$A_1^{3,3,2}:=\frac{1}{12\sqrt{14}}\big(-9S_{2a}^3\big(S_{3y}^2S_{3z}S_{4x}S_{4y}-S_{3x}S_{3y}^2S_{4y}S_{4z}+S_{3x}^2S_{4x}\big(S_{3z}S_{4y}$$
$$-S_{3y}S_{4z}\big)\big)-9S_{2a}S_{2b}^2\big(S_{3y}^2S_{3z}S_{4x}S_{4y}-S_{3x}S_{3y}^2S_{4y}S_{4z}+S_{3x}^2S_{4x}\big(S_{3z}S_{4y}$$

$$-S_{3y}S_{4z})\big) - 9\sqrt{3}S_{2a}^2S_{2b}\big(-S_{3y}^2S_{3z}S_{4x}S_{4y} + S_{3x}S_{3y}^2S_{4y}S_{4z} + S_{3x}^2S_{4x}(S_{3z}S_{4y} - S_{3y}S_{4z})\big) + \sqrt{3}S_{2b}^3\big(-S_{3x}\big(S_{3y}^2 + 8S_{3z}^2\big)S_{4y}S_{4z} + S_{3y}S_{3z}S_{4x}(S_{3y}S_{4y} + 8S_{3z}S_{4z}) + S_{3x}^2(-S_{3z}S_{4x}S_{4y} + S_{3y}S_{4x}S_{4z})\big)\big)$$

$$A_1^{3,4,1} := \frac{1}{12\sqrt{14}}\big(-9S_{2a}^3\big(S_{3x}^3S_{3z}S_{4y} - S_{3x}^3S_{3y}S_{4z} + S_{3y}^3(S_{3z}S_{4x} - S_{3x}S_{4z})\big) -9S_{2a}S_{2b}^2\big(S_{3x}^3S_{3z}S_{4y} - S_{3x}^3S_{3y}S_{4z} + S_{3y}^3(S_{3z}S_{4x} - S_{3x}S_{4z})\big) +9\sqrt{3}S_{2a}^2S_{2b}\big(-S_{3x}^3S_{3z}S_{4y} + S_{3x}^3S_{3y}S_{4z} + S_{3y}^3(S_{3z}S_{4x} - S_{3x}S_{4z})\big) +\sqrt{3}S_{2b}^3\big(-S_{3z}\big(S_{3x}^3 + 8S_{3x}S_{3z}^2\big)S_{4y} + S_{3y}^3(S_{3z}S_{4x} - S_{3x}S_{4z}) +S_{3y}\big(8S_{3z}^3S_{4x} + S_{3x}^3S_{4z}\big)\big)\big)$$

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$$A_1^{0,4,5} := \frac{1}{\sqrt{6}}\big(S_{3x}^3S_{4x}^4(S_{3z}S_{4y} + S_{3y}S_{4z}) + S_{3y}S_{3z}S_{4x}\big(S_{3y}^2S_{4y}^4 + S_{3z}^2S_{4z}^4\big) +S_{3x}\big(S_{3y}^3S_{4y}^4S_{4z} + S_{3z}^3S_{4y}S_{4z}^4\big)\big)$$

$$A_1^{0,5,4} := \frac{1}{\sqrt{6}}\big(S_{3x}^4S_{4x}^3(S_{3z}S_{4y} + S_{3y}S_{4z}) + S_{3y}S_{3z}S_{4x}\big(S_{3y}^3S_{4y}^3 + S_{3z}^3S_{4z}^3\big) +S_{3x}\big(S_{3y}^4S_{4y}^3S_{4z} + S_{3z}^4S_{4y}S_{4z}^3\big)\big)$$

$$A_1^{1,2,6} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3y}^2S_{4y}^4\big(S_{4x}^2 - S_{4z}^2\big) + S_{3x}^2S_{4x}^4\big(S_{4y}^2 - S_{4z}^2\big)\big) +\sqrt{3}S_{2b}\big(2S_{3z}^2\big(-S_{4x}^2 + S_{4y}^2\big)S_{4z}^4 + S_{3x}^2S_{4x}^4\big(S_{4y}^2 - S_{4z}^2\big) +S_{3y}^2S_{4y}^4\big(-S_{4x}^2 + S_{4z}^2\big)\big)\big)$$

$$A_1^{1,3,5} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3y}^3S_{4y}^3\big(S_{4x}^2 - S_{4z}^2\big) + S_{3x}^3S_{4x}^3\big(S_{4y}^2 - S_{4z}^2\big)\big) +\sqrt{3}S_{2b}\big(2S_{3z}^3\big(-S_{4x}^2 + S_{4y}^2\big)S_{4z}^3 + S_{3x}^3S_{4x}^3\big(S_{4y}^2 - S_{4z}^2\big) +S_{3y}^3S_{4y}^3\big(-S_{4x}^2 + S_{4z}^2\big)\big)\big)$$

$$A_2^{1,3,5} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3x}S_{3y}^2S_{4x}S_{4y}^4 - S_{3y}^2S_{3z}S_{4y}^4S_{4z} + S_{3x}^2S_{4x}^4(S_{3y}S_{4y} - S_{3z}S_{4z})\big) +\sqrt{3}S_{2b}\big(S_{3x}^2S_{4x}^4(S_{3y}S_{4y} - S_{3z}S_{4z}) + S_{3y}S_{3z}S_{4y}S_{4z}\big(S_{3y}S_{4y}^3 + 2S_{3z}S_{4z}^3\big) -S_{3x}S_{4x}\big(S_{3y}^2S_{4y}^4 + 2S_{3z}^2S_{4z}^4\big)\big)\big)$$

$$A_1^{1,4,4} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3y}^4S_{4y}^2\big(S_{4x}^2 - S_{4z}^2\big) + S_{3x}^4S_{4x}^2\big(S_{4y}^2 - S_{4z}^2\big)\big) +\sqrt{3}S_{2b}\big(2S_{3z}^4\big(-S_{4x}^2 + S_{4y}^2\big)S_{4z}^2 + S_{3x}^4S_{4x}^2\big(S_{4y}^2 - S_{4z}^2\big) +S_{3y}^4S_{4y}^2\big(-S_{4x}^2 + S_{4z}^2\big)\big)\big)$$

$$A_2^{1,4,4} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3x}S_{3y}^3S_{4x}S_{4y}^3 - S_{3y}^3S_{3z}S_{4y}^3S_{4z} + S_{3x}^3S_{4x}^3(S_{3y}S_{4y} - S_{3z}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}\big(S_{3y}^3S_{3z}S_{4y}^3S_{4z} + 2S_{3y}S_{3z}^3S_{4y}S_{4z}^3 + S_{3x}^3S_{4x}^3(S_{3y}S_{4y} - S_{3z}S_{4z})$$
$$-S_{3x}S_{4x}\big(S_{3y}^3S_{4y}^3 + 2S_{3z}^3S_{4z}^3\big)\big)\big)$$

$$A_1^{1,5,3} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3x}S_{3y}^4S_{4x}S_{4y}^2 - S_{3y}^4S_{3z}S_{4y}^2S_{4z} + S_{3x}^4S_{4x}^2(S_{3y}S_{4y} - S_{3z}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}\big(S_{3x}^4S_{4x}^2(S_{3y}S_{4y} - S_{3z}S_{4z}) + S_{3y}S_{3z}S_{4y}S_{4z}\big(S_{3y}^3S_{4y} + 2S_{3z}^3S_{4z}\big)$$
$$-S_{3x}S_{4x}\big(S_{3y}^4S_{4y}^2 + 2S_{3z}^4S_{4z}^2\big)\big)\big)$$

$$A_2^{1,5,3} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3x}S_{3y}^4S_{4x}^3 - S_{3y}^4S_{3z}S_{4z}^3 + S_{3x}^4\big(S_{3y}S_{4y}^3 - S_{3z}S_{4z}^3\big)\big)$$
$$+\sqrt{3}S_{2b}\big(-S_{3x}\big(S_{3y}^4 + 2S_{3z}^4\big)S_{4x}^3 + 2S_{3y}S_{3z}^4S_{4y}^3 + S_{3y}^4S_{3z}S_{4z}^3$$
$$+S_{3x}^4\big(S_{3y}S_{4y}^3 - S_{3z}S_{4z}^3\big)\big)\big)$$

$$A_1^{1,6,2} := \frac{1}{6\sqrt{2}}\big(3S_{2a}\big(S_{3x}^4\big(S_{3y}^2 - S_{3z}^2\big)S_{4x}^2 + S_{3x}^2S_{3y}^4S_{4y}^2 - S_{3y}^4S_{3z}^2S_{4y}^2\big)$$
$$+\sqrt{3}S_{2b}\big(S_{3x}^4\big(S_{3y}^2 - S_{3z}^2\big)S_{4x}^2 + S_{3y}^4S_{3z}^2S_{4y}^2 + 2S_{3y}^2S_{3z}^4S_{4z}^2$$
$$-S_{3x}^2\big(S_{3y}^4S_{4y}^2 + 2S_{3z}^4S_{4z}^2\big)\big)\big)$$

$$A_1^{2,2,5} := \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3y}S_{3z}S_{4x}S_{4y}^4 + S_{3x}\big(S_{3z}S_{4x}^4S_{4y} + S_{3y}\big(S_{4x}^4 - S_{4y}^4\big)S_{4z}\big)\big)$$
$$+3\sqrt{3}S_{2a}^2\big(S_{3y}S_{3z}S_{4x}S_{4y}^4 + S_{3x}\big(S_{3z}S_{4x}^4S_{4y} + S_{3y}\big(S_{4x}^4 + S_{4y}^4\big)S_{4z}\big)\big)$$
$$+\sqrt{3}S_{2b}^2\big(S_{3y}S_{3z}S_{4x}\big(S_{4y}^4 + 4S_{4z}^4\big) + S_{3x}\big(S_{3y}\big(S_{4x}^4 + S_{4y}^4\big)S_{4z}$$
$$+S_{3z}S_{4y}\big(S_{4x}^4 + 4S_{4z}^4\big)\big)\big)\big)$$

$$A_1^{2,3,4} := \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3y}^2S_{3z}S_{4x}S_{4y}^3 - S_{3x}S_{3y}^2S_{4y}^3S_{4z} + S_{3x}^2S_{4x}^3(S_{3z}S_{4y} + S_{3y}S_{4z})\big)$$
$$+3\sqrt{3}S_{2a}^2\big(S_{3y}^2S_{3z}S_{4x}S_{4y}^3 + S_{3x}S_{3y}^2S_{4y}^3S_{4z} + S_{3x}^2S_{4x}^3(S_{3z}S_{4y} + S_{3y}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}^2\big(S_{3x}^2S_{4x}^3(S_{3z}S_{4y} + S_{3y}S_{4z}) + S_{3y}S_{3z}S_{4x}\big(S_{3y}S_{4y}^3 + 4S_{3z}S_{4z}^3\big)$$
$$+S_{3x}\big(S_{3y}^2S_{4y}^3S_{4z} + 4S_{3z}^2S_{4y}S_{4z}^3\big)\big)\big)$$

$$A_2^{2,3,4} := \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}S_{4z}\big(-S_{3x}^2S_{3y}S_{4x}^3 + S_{3x}S_{3y}^2S_{4y}^3 - S_{3y}S_{3z}^2S_{4x}S_{4z}^2 + S_{3x}S_{3z}^2S_{4y}S_{4z}^2\big)$$
$$+3\sqrt{3}S_{2a}^2S_{4z}\big(S_{3x}^2S_{3y}S_{4x}^3 + S_{3x}S_{3y}^2S_{4y}^3 + S_{3y}S_{3z}^2S_{4x}S_{4z}^2 + S_{3x}S_{3z}^2S_{4y}S_{4z}^2\big)$$
$$+\sqrt{3}S_{2b}^2\big(S_{3x}^2S_{4x}^3(4S_{3z}S_{4y} + S_{3y}S_{4z}) + S_{3y}S_{3z}S_{4x}\big(4S_{3y}S_{4y}^3 + S_{3z}S_{4z}^3\big)$$
$$+S_{3x}\big(S_{3y}^2S_{4y}^3S_{4z} + S_{3z}^2S_{4y}S_{4z}^3\big)\big)\big)$$

$$A_1^{2,4,3} := \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3y}^3S_{3z}S_{4x}S_{4y}^2 - S_{3x}S_{3y}^3S_{4y}^2S_{4z}$$
$$+S_{3x}^3S_{4x}^2(S_{3z}S_{4y} + S_{3y}S_{4z})\big) + 3\sqrt{3}S_{2a}^2\big(S_{3y}^3S_{3z}S_{4x}S_{4y}^2$$
$$+S_{3x}S_{3y}^3S_{4y}^2S_{4z} + S_{3x}^3S_{4x}^2(S_{3z}S_{4y} + S_{3y}S_{4z})\big) + \sqrt{3}S_{2b}^2\big(S_{3x}^3S_{4x}^2(S_{3z}S_{4y}$$

$$
\begin{aligned}
&+S_{3y}S_{4z}) + S_{3x}S_{4y}S_{4z}\left(S_{3y}^3 S_{4y} + 4S_{3z}^3 S_{4z}\right)\\
&+S_{3y}S_{3z}S_{4x}\left(S_{3y}^2 S_{4y}^2 + 4S_{3z}^2 S_{4z}^2\right)))\\
A_2^{2,4,3} :=& \frac{1}{6\sqrt{10}}\left(6S_{2a}S_{2b}S_{4z}\left(-S_{3x}S_{3y}^3 S_{4x}^2 - S_{3y}^3 S_{3z}S_{4x}S_{4z} + S_{3x}^3 S_{4y}(S_{3y}S_{4y} + S_{3z}S_{4z})\right)\right.\\
&+3\sqrt{3}S_{2a}^2 S_{4z}\left(S_{3x}S_{3y}^3 S_{4x}^2 + S_{3y}^3 S_{3z}S_{4x}S_{4z} + S_{3x}^3 S_{4y}(S_{3y}S_{4y} + S_{3z}S_{4z})\right)\\
&+\sqrt{3}S_{2b}^2\left(S_{3x}S_{4x}^2\left(4S_{3z}^3 S_{4y} + S_{3y}^3 S_{4z}\right) + S_{3x}^3 S_{4y}S_{4z}(S_{3y}S_{4y} + S_{3z}S_{4z})\right.\\
&\left.\left.+S_{3y}S_{3z}S_{4x}\left(4S_{3z}^2 S_{4y}^2 + S_{3y}^2 S_{4z}^2\right)\right)\right)\\
A_1^{2,5,2} :=& \frac{1}{6\sqrt{10}}\left(6S_{2a}S_{2b}\left(-S_{3y}^4 S_{3z}S_{4x}S_{4y} - S_{3x}S_{3y}^4 S_{4y}S_{4z} + S_{3x}^4 S_{4x}(S_{3z}S_{4y}\right.\right.\\
&+S_{3y}S_{4z})) + 3\sqrt{3}S_{2a}^2\left(S_{3y}^4 S_{3z}S_{4x}S_{4y} + S_{3x}S_{3y}^4 S_{4y}S_{4z} + S_{3x}^4 S_{4x}(S_{3z}S_{4y}\right.\\
&+S_{3y}S_{4z})) + \sqrt{3}S_{2b}^2\left(S_{3x}\left(S_{3y}^4 + 4S_{3z}^4\right)S_{4y}S_{4z} + S_{3x}^4 S_{4x}(S_{3z}S_{4y}\right.\\
&+S_{3y}S_{4z}) + S_{3y}S_{3z}S_{4x}\left(S_{3y}^3 S_{4y} + 4S_{3z}^3 S_{4z}\right)))\\
A_1^{3,0,6} :=& \frac{1}{12\sqrt{14}}\left(9S_{2a}^3\left(S_{4x}^2 S_{4y}^4 - S_{4y}^4 S_{4z}^2 + S_{4x}^4\left(S_{4y}^2 - S_{4z}^2\right)\right)\right.\\
&+9S_{2a}S_{2b}^2\left(S_{4x}^2 S_{4y}^4 - S_{4y}^4 S_{4z}^2 + S_{4x}^4\left(S_{4y}^2 - S_{4z}^2\right)\right)\\
&+9\sqrt{3}S_{2a}^2 S_{2b}\left(-S_{4x}^2 S_{4y}^4 + S_{4y}^4 S_{4z}^2 + S_{4x}^4\left(S_{4y}^2 - S_{4z}^2\right)\right)\\
&\left.+\sqrt{3}S_{2b}^3\left(S_{4x}^4\left(S_{4y}^2 - S_{4z}^2\right) + S_{4y}^2 S_{4z}^2\left(S_{4y}^2 + 8S_{4z}^2\right) - S_{4x}^2\left(S_{4y}^4 + 8S_{4z}^4\right)\right)\right)\\
A_1^{3,1,5} :=& \frac{1}{12\sqrt{14}}\left(9S_{2a}^3\left(S_{3y}S_{4y}^3\left(S_{4x}^2 - S_{4z}^2\right) + S_{3x}S_{4x}^3\left(S_{4y}^2 - S_{4z}^2\right)\right)\right.\\
&+9S_{2a}S_{2b}^2\left(S_{3y}S_{4y}^3\left(S_{4x}^2 - S_{4z}^2\right) + S_{3x}S_{4x}^3\left(S_{4y}^2 - S_{4z}^2\right)\right.\\
&+9\sqrt{3}S_{2a}^2 S_{2b}\left(S_{3x}S_{4x}^3\left(S_{4y}^2 - S_{4z}^2\right) + S_{3y}S_{4y}^3\left(-S_{4x}^2 + S_{4z}^2\right)\right)\\
&+\sqrt{3}S_{2b}^3\left(8S_{3z}\left(-S_{4x}^2 + S_{4y}^2\right)S_{4z}^3 + S_{3x}S_{4x}^3\left(S_{4y}^2 - S_{4z}^2\right)\right.\\
&\left.+S_{3y}S_{4y}^3\left(-S_{4x}^2 + S_{4z}^2\right)\right)\\
A_1^{3,2,4} :=& \frac{1}{12\sqrt{14}}\left(9S_{2a}^3\left(S_{3y}^2 S_{4y}^2\left(S_{4x}^2 - S_{4z}^2\right) + S_{3x}^2 S_{4x}^2\left(S_{4y}^2 - S_{4z}^2\right)\right) + 9S_{2a}S_{2b}^2\right.\\
&\times\left(S_{3y}^2 S_{4y}^2\left(S_{4x}^2 - S_{4z}^2\right) + S_{3x}^2 S_{4x}^2\left(S_{4y}^2 - S_{4z}^2\right)\right) + 9\sqrt{3}S_{2a}^2 S_{2b}\left(S_{3x}^2 S_{4x}^2\right.\\
&\times\left(S_{4y}^2 - S_{4z}^2\right) + S_{3y}^2 S_{4y}^2\left(-S_{4x}^2 + S_{4z}^2\right) + \sqrt{3}S_{2b}^3\left(8S_{3z}^2\left(-S_{4x}^2 + S_{4y}^2\right)\right.\\
&\left.\times S_{4z}^2 + S_{3x}^2 S_{4x}^2\left(S_{4y}^2 - S_{4z}^2\right) + S_{3y}^2 S_{4y}^2\left(-S_{4x}^2 + S_{4z}^2\right)\right)))\\
A_2^{3,2,4} :=& \frac{1}{12\sqrt{14}}\left(9\sqrt{3}S_{2a}^2 S_{2b}\left(S_{3y}S_{3z}S_{4y}^3 S_{4z} + S_{3x}\left(S_{3y}S_{4x}S_{4y}\left(S_{4x}^2 - S_{4y}^2\right)\right.\right.\right.\\
&-S_{3z}S_{4x}^3 S_{4z})) + 9S_{2a}^3\left(-S_{3y}S_{3z}S_{4y}^3 S_{4z} + S_{3x}\left(S_{3y}S_{4x}S_{4y}\left(S_{4x}^2 + S_{4y}^2\right)\right.\right.\\
&-S_{3z}S_{4x}^3 S_{4z})) + 9S_{2a}S_{2b}^2\left(-S_{3y}S_{3z}S_{4y}^3 S_{4z} + S_{3x}\left(S_{3y}S_{4x}S_{4y}\left(S_{4x}^2 + S_{4y}^2\right)\right.\right.\\
&-S_{3z}S_{4x}^3 S_{4z})) + \sqrt{3}S_{2b}^3\left(S_{3y}S_{3z}S_{4y}S_{4z}\left(S_{4y}^2 + 8S_{4z}^2\right)\right)\\
&+S_{3x}S_{4x}\left(S_{3y}S_{4y}\left(S_{4x}^2 - S_{4y}^2\right) - S_{3z}S_{4z}\left(S_{4x}^2 + 8S_{4z}^2\right)\right))))
\end{aligned}
$$

$$A_1^{3,3,3} := \frac{1}{12\sqrt{14}}\big(9S_{2a}^3\big(S_{3y}^3 S_{4y}\big(S_{4x}^2 - S_{4z}^2\big) + S_{3x}^3 S_{4x}\big(S_{4y}^2 - S_{4z}^2\big)\big)$$
$$+9S_{2a}S_{2b}^2\big(S_{3y}^3 S_{4y}\big(S_{4x}^2 - S_{4z}^2\big) + S_{3x}^3 S_{4x}\big(S_{4y}^2 - S_{4z}^2\big)\big)$$
$$+\sqrt{3}S_{2b}^3\big(8S_{3z}^3\big(-S_{4x}^2 + S_{4y}^2\big)S_{4z} + S_{3x}^3 S_{4x}\big(S_{4y}^2 - S_{4z}^2\big)$$
$$+S_{3y}^3 S_{4y}\big(-S_{4x}^2 + S_{4z}^2\big) - 9\sqrt{3}S_{2a}^2 S_{2b}\big(S_{3y}^3 S_{4y}\big(S_{4x}^2 - S_{4z}^2\big)$$
$$+S_{3x}^3 S_{4x}\big(-S_{4y}^2 + S_{4z}^2\big)\big)\big)$$

$$A_2^{3,3,3} := \frac{1}{12\sqrt{14}}\big(9\sqrt{3}S_{2a}^2 S_{2b}(S_{3x}S_{4x} - S_{3y}S_{4y})\big(-S_{3y}S_{3z}S_{4y}S_{4z} + S_{3x}S_{4x}$$
$$(S_{3y}S_{4y} - S_{3z}S_{4z})\big) + 9S_{2a}^3\big(S_{3x}S_{3y}^2 S_{4x}S_{4y}^2 - S_{3y}^2 S_{3z}S_{4y}^2 S_{4z} + S_{3x}^2 S_{4x}^2$$
$$(S_{3y}S_{4y} - S_{3z}S_{4z})\big) + 9S_{2a}S_{2b}^2\big(S_{3x}S_{3y}^2 S_{4x}S_{4y}^2 - S_{3y}^2 S_{3z}S_{4y}^2 S_{4z} + S_{3x}^2 S_{4x}^2$$
$$\big(S_{3y}S_{4y} - S_{3z}S_{4z}\big)\big) + \sqrt{3}S_{2b}^3\big(S_{3x}^2 S_{4x}^2(S_{3y}S_{4y} - S_{3z}S_{4z}) + S_{3y}S_{3z}S_{4y}S_{4z}$$
$$(S_{3y}S_{4y} + 8S_{3z}S_{4z}) - S_{3x}S_{4x}\big(S_{3y}^2 S_{4y}^2 + 8S_{3z}^2 S_{4z}^2\big)\big)\big)$$

$$A_1^{3,4,2} := \frac{1}{12\sqrt{14}}\big(9S_{2a}^3\big(S_{3y}^4\big(S_{4x}^2 - S_{4z}^2\big) + S_{3x}^4\big(S_{4y}^2 - S_{4z}^2\big)\big) + 9S_{2a}S_{2b}^2$$
$$\big(S_{3y}^4\big(S_{4x}^2 - S_{4z}^2\big) + S_{3x}^4\big(S_{4y}^2 - S_{4z}^2\big)\big) + \sqrt{3}S_{2b}^3\big(-8S_{3z}^4\big(S_{4x}^2 - S_{4y}^2\big)$$
$$+S_{3x}^4\big(S_{4y}^2 - S_{4z}^2\big) + S_{3y}^4\big(-S_{4x}^2 + S_{4z}^2\big)\big) - 9\sqrt{3}S_{2a}^2 S_{2b}\big(S_{3y}^4\big(S_{4x}^2 - S_{4z}^2\big)$$
$$+S_{3x}^4\big(-S_{4y}^2 + S_{4z}^2\big)\big)\big)$$

$$A_2^{3,4,2} := \frac{1}{12\sqrt{14}}\big(9S_{2a}^3\big(S_{3x}S_{3y}^3 S_{4x}S_{4y} - S_{3y}^3 S_{3z}S_{4y}S_{4z} + S_{3x}^3 S_{4x}(S_{3y}S_{4y} - S_{3z}S_{4z})\big)$$
$$+9S_{2a}S_{2b}^2\big(S_{3x}S_{3y}^3 S_{4x}S_{4y} - S_{3y}^3 S_{3z}S_{4y}S_{4z} + S_{3x}^3 S_{4x}(S_{3y}S_{4y} - S_{3z}S_{4z})\big)$$
$$+9\sqrt{3}S_{2a}^2 S_{2b}\big(-S_{3x}S_{3y}^3 S_{4x}S_{4y} + S_{3y}^3 S_{3z}S_{4y}S_{4z} + S_{3x}^3 S_{4x}\big(S_{3y}S_{4y}$$
$$\big(S_{3y}S_{4y} - S_{3z}S_{4z}\big)\big) + \sqrt{3}S_{2b}^3\big(S_{3y}S_{3z}\big(S_{3y}^2 + 8S_{3z}^2\big)S_{4y}S_{4z} + S_{3x}^3 S_{4x}$$
$$(S_{3y}S_{4y} - S_{3z}S_{4z}) - S_{3x}S_{4x}\big(S_{3y}^3 S_{4y} + 8S_{3z}^3 S_{4z}\big)$$

$$A_1^{3,5,1} := \frac{1}{12\sqrt{14}}\big(9S_{2a}^3\big(S_{3x}S_{3y}^4 S_{4x} - S_{3y}^4 S_{3z}S_{4z} + S_{3x}^4(S_{3y}S_{4y} - S_{3z}S_{4z})\big)$$
$$+9S_{2a}S_{2b}^2\big(S_{3x}S_{3y}^4 S_{4x} - S_{3y}^4 S_{3z}S_{4z} + S_{3x}^4(S_{3y}S_{4y} - S_{3z}S_{4z})\big)$$
$$+9\sqrt{3}S_{2a}^2 S_{2b}\big(-S_{3x}S_{3y}^4 S_{4x} + S_{3y}^4 S_{3z}S_{4z} + S_{3x}^4(S_{3y}S_{4y} - S_{3z}S_{4z})\big)$$
$$+\sqrt{3}S_{2b}^3\big(-S_{3x}\big(S_{3y}^4 + 8S_{3z}^4\big)S_{4x} + 8S_{3y}S_{3z}^4 S_{4y} + S_{3y}^4 S_{3z}S_{4z}$$
$$+S_{3x}^4(S_{3y}S_{4y} - S_{3z}S_{4z})\big)\big)$$

$$A_1^{3,6,0} := \frac{1}{12\sqrt{14}}\big(9S_{2a}^3\big(S_{3x}^2 S_{3y}^4 - S_{3y}^4 S_{3z}^2 + S_{3x}^4\big(S_{3y}^2 - S_{3z}^2\big)\big) + 9S_{2a}S_{2b}^2\big(S_{3x}^2 S_{3y}^4$$
$$-S_{3y}^4 S_{3z}^2 + S_{3x}^4\big(S_{3y}^2 - S_{3z}^2\big)\big) + 9\sqrt{3}S_{2a}^2 S_{2b}\big(-S_{3x}^2 S_{3y}^4 + S_{3y}^4 S_{3z}^2$$
$$+S_{3x}^4\big(S_{3y}^2 - S_{3z}^2\big)\big) + \sqrt{3}S_{2b}^3\big(S_{3x}^4\big(S_{3y}^2 - S_{3z}^2\big) + S_{3y}^2 S_{3z}^2\big(S_{3y}^2 + 8S_{3z}^2\big)$$
$$-S_{3x}^2\big(S_{3y}^4 + 8S_{3z}^4\big)\big)\big)$$

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$$A_1^{0,5,5} := \frac{1}{\sqrt{6}}\big(S_{3y}^4 S_{3z} S_{4y}^4 S_{4z} + S_{3y} S_{3z}^4 S_{4y} S_{4z}^4 + S_{3x}^4 S_{4x}^4 (S_{3y} S_{4y} + S_{3z} S_{4z}) + S_{3x} S_{4x}\big(S_{3y}^4 S_{4y}^4 + S_{3z}^4 S_{4z}^4\big)\big)$$

$$A_1^{1,4,5} := \frac{1}{6\sqrt{2}}\big(-3S_{2a}\big(S_{3y}^3 S_{3z} S_{4x} S_{4y}^4 - S_{3x} S_{3y}^3 S_{4y}^4 S_{4z} + S_{3x}^3 S_{4x}^4 (S_{3z} S_{4y} - S_{3y} S_{4z})\big) + \sqrt{3}S_{2b}\big(S_{3x}^3 S_{4x}^4 (-S_{3z} S_{4y} + S_{3y} S_{4z}) + S_{3y} S_{3z} S_{4x}\big(S_{3y}^2 S_{4y}^4 + 2S_{3z}^2 S_{4z}^4\big) - S_{3x}\big(S_{3y}^3 S_{4y}^4 S_{4z} + 2S_{3z}^3 S_{4y} S_{4z}^4\big)\big)\big)$$

$$A_1^{1,5,4} := \frac{1}{6\sqrt{2}}\big(-3S_{2a}\big(S_{3y}^4 S_{3z} S_{4x} S_{4y}^3 - S_{3x} S_{3y}^4 S_{4y}^3 S_{4z} + S_{3x}^4 S_{4x}^3 (S_{3z} S_{4y} - S_{3y} S_{4z})\big) + \sqrt{3}S_{2b}\big(S_{3x}^4 S_{4x}^3 (-S_{3z} S_{4y} + S_{3y} S_{4z}) + S_{3y} S_{3z} S_{4x}\big(S_{3y}^3 S_{4y}^3 + 2S_{3z}^3 S_{4z}^3\big) - S_{3x}\big(S_{3y}^4 S_{4y}^3 S_{4z} + 2S_{3z}^4 S_{4y} S_{4z}^3\big)\big)\big)$$

$$A_1^{2,2,6} := \frac{1}{6\sqrt{10}}\big(6S_{2a} S_{2b}\big(S_{3y}^2 S_{4x}^4 S_{4y}^2 - S_{3x}^2 S_{4x}^2 S_{4y}^4 + S_{3z}^2\big(S_{4x}^4 - S_{4y}^4\big) S_{4z}^2\big) + 3\sqrt{3}S_{2a}^2\big(S_{3y}^2 S_{4x}^4 S_{4y}^2 + S_{3x}^2 S_{4x}^2 S_{4y}^4 + S_{3z}^2\big(S_{4x}^4 + S_{4y}^4\big) S_{4z}^2\big) + \sqrt{3}S_{2b}^2\big(S_{3z}^2\big(S_{4x}^4 + S_{4y}^4\big) S_{4z}^2 + S_{3y}^2 S_{4y}^2\big(S_{4x}^4 + 4S_{4z}^4\big) + S_{3x}^2 S_{4x}^2\big(S_{4y}^4 + 4S_{4z}^4\big)\big)\big)$$

$$A_1^{2,3,5} := \frac{1}{6\sqrt{10}}\big(6S_{2a} S_{2b}\big(-S_{3x} S_{3y}^2 S_{4x} S_{4y}^4 - S_{3y}^2 S_{3z} S_{4y}^4 S_{4z} + S_{3x}^2 S_{4x}^4 (S_{3y} S_{4y} + S_{3z} S_{4z})\big) + 3\sqrt{3}S_{2a}^2\big(S_{3x} S_{3y}^2 S_{4x} S_{4y}^4 + S_{3y}^2 S_{3z} S_{4y}^4 S_{4z} + S_{3x}^2 S_{4x}^4 (S_{3y} S_{4y} + S_{3z} S_{4z})\big) + \sqrt{3}S_{2b}^2\big(S_{3x}^2 S_{4x}^4 (S_{3y} S_{4y} + S_{3z} S_{4z}) + S_{3y} S_{3z} S_{4y} S_{4z}\big(S_{3y} S_{4y}^3 + 4S_{3z} S_{4z}^3\big) + S_{3x} S_{4x}\big(S_{3y}^2 S_{4y}^4 + 4S_{3z}^2 S_{4z}^4\big)\big)\big)$$

$$A_2^{2,3,5} := \frac{1}{6\sqrt{10}}\big(-6S_{2a} S_{2b}\big(S_{3x}^2 S_{3y} S_{4x}^2 S_{4y}^3 + S_{3y} S_{3z}^2 S_{4y}^3 S_{4z}^2 - S_{3x} S_{4x}^3\big(S_{3y}^2 S_{4y}^2 + S_{3z}^2 S_{4z}^2\big)\big) + 3\sqrt{3}S_{2a}^2\big(S_{3x}^2 S_{3y} S_{4x}^2 S_{4y}^3 + S_{3y} S_{3z}^2 S_{4y}^3 S_{4z}^2 + S_{3x} S_{4x}^3\big(S_{3y}^2 S_{4y}^2 + S_{3z}^2 S_{4z}^2\big)\big) + \sqrt{3}S_{2b}^2\big(S_{3y} S_{3z} S_{4y}^2 S_{4z}^2 (S_{3z} S_{4y} + 4S_{3y} S_{4z}) + S_{3x} S_{4x}^3\big(S_{3y}^2 S_{4y}^2 + S_{3z}^2 S_{4z}^2\big) + S_{3x}^2 S_{4x}^2\big(S_{3y} S_{4y}^3 + 4S_{3z} S_{4z}^3\big)\big)\big)$$

$$A_1^{2,4,4} := \frac{1}{6\sqrt{5}}\big(6S_{2a} S_{2b}\big(S_{3x}^4 S_{4x}^4 - S_{3y}^4 S_{4y}^4\big) + 3\sqrt{3}S_{2a}^2\big(S_{3x}^4 S_{4x}^4 + S_{3y}^4 S_{4y}^4\big) + \sqrt{3}S_{2b}^2\big(S_{3x}^4 S_{4x}^4 + S_{3y}^4 S_{4y}^4 + 4S_{3z}^4 S_{4z}^4\big)\big)$$

$$A_2^{2,4,4} := \frac{1}{6\sqrt{10}}\big(6S_{2a} S_{2b}\big(-S_{3x} S_{3y}^3 S_{4x} S_{4y}^3 - S_{3y}^3 S_{3z} S_{4y}^3 S_{4z} + S_{3x}^3 S_{4x}^3\big(S_{3y} S_{4y} + S_{3z} S_{4z}\big)\big) + 3\sqrt{3}S_{2a}^2\big(S_{3x} S_{3y}^3 S_{4x} S_{4y}^3 + S_{3y}^3 S_{3z} S_{4y}^3 S_{4z} + S_{3x}^3 S_{4x}^3\big(S_{3y} S_{4y}$$

$$
\begin{aligned}
&+S_{3z}S_{4z})\big) + \sqrt{3}S_{2b}^2\big(S_{3y}^3S_{3z}S_{4y}^3S_{4z} + 4S_{3y}S_{3z}^3S_{4y}S_{4z}^3 + S_{3x}^3S_{4x}^3\big(S_{3y}S_{4y}\\
&+S_{3z}S_{4z}\big) + S_{3x}S_{4x}\big(S_{3y}^3S_{4y}^3 + 4S_{3z}^3S_{4z}^3\big)\big)\big)\\
A_1^{2,5,3} := {}& \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3x}S_{3y}^4S_{4x}S_{4y}^2 - S_{3y}^4S_{3z}S_{4y}^2S_{4z} + S_{3x}^4S_{4x}^2\big(S_{3y}S_{4y}\\
&+S_{3z}S_{4z})\big) + 3\sqrt{3}S_{2a}^2\big(S_{3x}S_{3y}^4S_{4x}S_{4y}^2 + S_{3y}^4S_{3z}S_{4y}^2S_{4z} + S_{3x}^4S_{4x}^2\big(S_{3y}S_{4y}\\
&+S_{3z}S_{4z})\big) + \sqrt{3}S_{2b}^2\big(S_{3x}^4S_{4x}^2(S_{3y}S_{4y} + S_{3z}S_{4z}) + S_{3y}S_{3z}S_{4y}S_{4z}\big(S_{3y}^3S_{4y}\\
&+4S_{3z}^3S_{4z}\big) + S_{3x}S_{4x}\big(S_{3y}^4S_{4y}^2 + 4S_{3z}^4S_{4z}^2\big)\big)\big)\\
A_2^{2,5,3} := {}& \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3x}S_{3y}^4S_{4x}^3 - S_{3y}^4S_{3z}S_{4z}^3 + S_{3x}^4\big(S_{3y}S_{4y}^3 + S_{3z}S_{4z}^3\big)\big)\\
&+3\sqrt{3}S_{2a}^2\big(S_{3x}S_{3y}^4S_{4x}^3 + S_{3y}^4S_{3z}S_{4z}^3 + S_{3x}^4\big(S_{3y}S_{4y}^3 + S_{3z}S_{4z}^3\big)\big)\\
&+\sqrt{3}S_{2b}^2\big(S_{3x}\big(S_{3y}^4 + 4S_{3z}^4\big)S_{4x}^3 + 4S_{3y}S_{3z}^4S_{4y}^3 + S_{3y}^4S_{3z}S_{4z}^3\\
&+S_{3x}^4\big(S_{3y}S_{4y}^3 + S_{3z}S_{4z}^3\big)\big)\big)\\
A_1^{2,6,2} := {}& \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3x}^2S_{3y}^4S_{4x}^2 - S_{3y}^4S_{3z}^2S_{4z}^2 + S_{3x}^4\big(S_{3y}^2S_{4y}^2 + S_{3z}^2S_{4z}^2\big)\big)\\
&+3\sqrt{3}S_{2a}^2\big(S_{3x}^2S_{3y}^4S_{4x}^2 + S_{3y}^4S_{3z}^2S_{4z}^2 + S_{3x}^4\big(S_{3y}^2S_{4y}^2 + S_{3z}^2S_{4z}^2\big)\big)\\
&+\sqrt{3}S_{2b}^2\big(S_{3x}^2\big(S_{3y}^4 + 4S_{3z}^4\big)S_{4x}^2 + 4S_{3y}^2S_{3z}^4S_{4y}^2 + S_{3y}^4S_{3z}^2S_{4z}^2\\
&+S_{3x}^4\big(S_{3y}^2S_{4y}^2 + S_{3z}^2S_{4z}^2\big)\big)\big)\\
A_1^{3,2,5} := {}& \frac{1}{12\sqrt{14}}\big(9\sqrt{3}S_{2a}^2S_{2b}\big(S_{3y}S_{3z}S_{4x}S_{4y}^4 - S_{3x}\big(S_{3z}S_{4x}^4S_{4y} + S_{3y}\big(-S_{4x}^4\\
&+S_{4y}^4\big)S_{4z}\big)\big) - 9S_{2a}^3\big(S_{3y}S_{3z}S_{4x}S_{4y}^4 + S_{3x}\big(S_{3z}S_{4x}^4S_{4y}\\
&-S_{3y}\big(S_{4x}^4 + S_{4y}^4\big)S_{4z}\big)\big) - 9S_{2a}S_{2b}^2\big(S_{3y}S_{3z}S_{4x}S_{4y}^4 + S_{3x}\big(S_{3z}S_{4x}^4S_{4y}\\
&-S_{3y}\big(S_{4x}^4 + S_{4y}^4\big)S_{4z}\big)\big) + \sqrt{3}S_{2b}^3\big(S_{3y}S_{3z}S_{4x}\big(S_{4y}^4 + 8S_{4z}^4\big)\\
&-S_{3x}\big(S_{3y}\big(-S_{4x}^4 + S_{4y}^4\big)S_{4z} + S_{3z}S_{4y}\big(S_{4x}^4 + 8S_{4z}^4\big)\big)\big)\big)\\
A_1^{3,3,4} := {}& \frac{1}{12\sqrt{14}}\big(9S_{2a}^3S_{4z}\big(S_{3x}^3S_{4y}\big(S_{4y}^2 - S_{4z}^2\big) + S_{3y}^3\big(S_{4x}^3 - S_{4x}S_{4z}^2\big)\big)\\
&+9S_{2a}S_{2b}^2S_{4z}\big(S_{3x}^3S_{4y}\big(S_{4y}^2 - S_{4z}^2\big) + S_{3y}^3\big(S_{4x}^3 - S_{4x}S_{4z}^2\big)\big)\\
&-9\sqrt{3}S_{2a}^2S_{2b}S_{4z}\big(S_{3x}^3S_{4y}\big(-S_{4y}^2 + S_{4z}^2\big) + S_{3y}^3\big(S_{4x}^3 - S_{4x}S_{4z}^2\big)\big)\\
&+\sqrt{3}S_{2b}^3\big(S_{3z}^3\big(-8S_{4x}^3S_{4y} + 8S_{4x}S_{4y}^3\big) + S_{3x}^3S_{4y}S_{4z}\big(S_{4y}^2 - S_{4z}^2\big)\\
&+S_{3y}^3\big(-S_{4x}^3S_{4z} + S_{4x}S_{4z}^3\big)\big)\big)\\
A_1^{3,4,3} := {}& \frac{1}{12\sqrt{14}}\big(-9S_{2a}^3\big(S_{3y}^3S_{3z}S_{4x}S_{4y}^2 - S_{3x}S_{3y}^3S_{4y}^2S_{4z} + S_{3x}^3S_{4x}^2\big(S_{3z}S_{4y}\\
&-S_{3y}S_{4z})\big) - 9S_{2a}S_{2b}^2\big(S_{3y}^3S_{3z}S_{4x}S_{4y}^2 - S_{3x}S_{3y}^3S_{4y}^2S_{4z} + S_{3x}^3S_{4x}^2\big(S_{3z}S_{4y}\\
&-S_{3y}S_{4z})\big) - 9\sqrt{3}S_{2a}^2S_{2b}\big(-S_{3y}^3S_{3z}S_{4x}S_{4y}^2 + S_{3x}S_{3y}^3S_{4y}^2S_{4z}
\end{aligned}
$$

$$+S_{3x}^3 S_{4x}^2 (S_{3z} S_{4y} - S_{3y} S_{4z})) + \sqrt{3} S_{2b}^3 \big(S_{3x}^3 S_{4x}^2 (-S_{3z} S_{4y} + S_{3y} S_{4z}) - S_{3x} S_{4y} S_{4z} \big(S_{3y}^3 S_{4y} + 8 S_{3z}^3 S_{4z}\big) + S_{3y} S_{3z} S_{4x} \big(S_{3y}^2 S_{4y}^2 + 8 S_{3z}^2 S_{4z}^2\big)\big)\big)$$

$$A_1^{3,5,2} := \frac{1}{12\sqrt{14}} \big(-9 S_{2a}^3 \big(S_{3y}^4 S_{3z} S_{4x} S_{4y} - S_{3x} S_{3y}^4 S_{4y} S_{4z} + S_{3x}^4 S_{4x} \big(S_{3z} S_{4y} - S_{3y} S_{4z}\big)\big) - 9 S_{2a} S_{2b}^2 \big(S_{3y}^4 S_{3z} S_{4x} S_{4y} - S_{3x} S_{3y}^4 S_{4y} S_{4z} + S_{3x}^4 S_{4x} \big(S_{3z} S_{4y} - S_{3y} S_{4z}\big)\big) - 9\sqrt{3} S_{2a}^2 S_{2b} \big(-S_{3y}^4 S_{3z} S_{4x} S_{4y} + S_{3x} S_{3y}^4 S_{4y} S_{4z} + S_{3x}^4 S_{4x} (S_{3z} S_{4y} - S_{3y} S_{4z})\big) + \sqrt{3} S_{2b}^3 \big(-S_{3x} \big(S_{3y}^4 + 8 S_{3z}^4\big) S_{4y} S_{4z} + S_{3y} S_{3z} S_{4x} \big(S_{3y}^3 S_{4y} + 8 S_{3z}^3 S_{4z}\big) + S_{3x}^4 (-S_{3z} S_{4x} S_{4y} + S_{3y} S_{4x} S_{4z})\big)\big)$$

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$$A_1^{1,4,6} := \frac{1}{6\sqrt{2}} \big(3 S_{2a} \big(S_{3y}^4 S_{4y}^4 \big(S_{4x}^2 - S_{4z}^2\big) + S_{3x}^4 S_{4x}^4 \big(S_{4y}^2 - S_{4z}^2\big)\big) + \sqrt{3} S_{2b} \big(2 S_{3z}^4 \big(-S_{4x}^2 + S_{4y}^2\big) S_{4z}^4 + S_{3x}^4 S_{4x}^4 \big(S_{4y}^2 - S_{4z}^2\big) + S_{3y}^4 S_{4y}^4 \big(-S_{4x}^2 + S_{4z}^2\big)\big)\big)$$

$$A_1^{1,5,5} := \frac{1}{6\sqrt{2}} \big(3 S_{2a} \big(S_{3x} S_{3y}^4 S_{4x} S_{4y}^4 - S_{3y}^4 S_{3z} S_{4y}^4 S_{4z} + S_{3x}^4 S_{4x}^4 (S_{3y} S_{4y} - S_{3z} S_{4z})\big) + \sqrt{3} S_{2b} \big(S_{3y}^4 S_{3z} S_{4y}^4 S_{4z} + 2 S_{3y} S_{3z}^4 S_{4y} S_{4z}^4 + S_{3x}^4 S_{4x}^4 (S_{3y} S_{4y} - S_{3z} S_{4z}) - S_{3x} S_{4x} \big(S_{3y}^4 S_{4y}^4 + 2 S_{3z}^4 S_{4z}^4\big)\big)\big)$$

$$A_1^{1,6,4} := \frac{1}{6\sqrt{2}} \big(3 S_{2a} \big(S_{3x}^4 \big(S_{3y}^2 - S_{3z}^2\big) S_{4x}^4 + S_{3x}^2 S_{3y}^4 S_{4y}^4 - S_{3y}^4 S_{3z}^2 S_{4y}^4\big) + \sqrt{3} S_{2b} \big(S_{3x}^4 \big(S_{3y}^2 - S_{3z}^2\big) S_{4x}^4 + S_{3y}^4 S_{3z}^2 S_{4y}^4 + 2 S_{3y}^2 S_{3z}^4 S_{4z}^4 - S_{3x}^2 \big(S_{3y}^4 S_{4y}^4 + 2 S_{3z}^4 S_{4z}^4\big)\big)\big)$$

$$A_1^{2,4,5} := \frac{1}{6\sqrt{10}} \big(6 S_{2a} S_{2b} \big(-S_{3y}^3 S_{3z} S_{4x} S_{4y}^4 - S_{3x} S_{3y}^3 S_{4y}^4 S_{4z} + S_{3x}^3 S_{4x}^4 \big(S_{3z} S_{4y} + S_{3y} S_{4z}\big)\big) + 3\sqrt{3} S_{2a}^2 \big(S_{3y}^3 S_{3z} S_{4x} S_{4y}^4 + S_{3x} S_{3y}^3 S_{4y}^4 S_{4z} + S_{3x}^3 S_{4x}^4 \big(S_{3z} S_{4y} + S_{3y} S_{4z}\big)\big) + \sqrt{3} S_{2b}^2 \big(S_{3x}^3 S_{4x}^4 (S_{3z} S_{4y} + S_{3y} S_{4z}) + S_{3y} S_{3z} S_{4x} \big(S_{3y}^2 S_{4y}^4 + 4 S_{3z}^2 S_{4z}^4\big) + S_{3x} \big(S_{3y}^3 S_{4y}^4 S_{4z} + 4 S_{3z}^3 S_{4y} S_{4z}^4\big)\big)\big)$$

$$A_1^{2,5,4} := \frac{1}{6\sqrt{10}} \big(6 S_{2a} S_{2b} \big(-S_{3y}^4 S_{3z} S_{4x} S_{4y}^3 - S_{3x} S_{3y}^4 S_{4y}^3 S_{4z} + S_{3x}^4 S_{4x}^3 \big(S_{3z} S_{4y} + S_{3y} S_{4z}\big)\big) + 3\sqrt{3} S_{2a}^2 \big(S_{3y}^4 S_{3z} S_{4x} S_{4y}^3 + S_{3x} S_{3y}^4 S_{4y}^3 S_{4z} + S_{3x}^4 S_{4x}^3 \big(S_{3z} S_{4y} + S_{3y} S_{4z}\big)\big) + \sqrt{3} S_{2b}^2 \big(S_{3x}^4 S_{4x}^3 (S_{3z} S_{4y} + S_{3y} S_{4z}) + S_{3y} S_{3z} S_{4x} \big(S_{3y}^3 S_{4y}^3 + 4 S_{3z}^3 S_{4z}^3\big) + S_{3x} \big(S_{3y}^4 S_{4y}^3 S_{4z} + 4 S_{3z}^4 S_{4y} S_{4z}^3\big)\big)\big)$$

$$
\begin{aligned}
A_1^{3,3,5} := \frac{1}{12\sqrt{14}}&\big(9S_{2a}^3\big(S_{3y}^3S_{4y}^3\big(S_{4x}^2-S_{4z}^2\big)+S_{3x}^3S_{4x}^3\big(S_{4y}^2-S_{4z}^2\big)\big)\\
&+9S_{2a}S_{2b}^2\big(S_{3y}^3S_{4y}^3\big(S_{4x}^2-S_{4z}^2\big)+S_{3x}^3S_{4x}^3\big(S_{4y}^2-S_{4z}^2\big)\big)\\
&+9\sqrt{3}S_{2a}^2S_{2b}\big(S_{3x}^3S_{4x}^3\big(S_{4y}^2-S_{4z}^2\big)+S_{3y}^3S_{4y}^3\big(-S_{4x}^2+S_{4z}^2\big)\big)\\
&+\sqrt{3}S_{2b}^3\big(8S_{3z}^3\big(-S_{4x}^2+S_{4y}^2\big)S_{4z}^3+S_{3x}^3S_{4x}^3\big(S_{4y}^2-S_{4z}^2\big)\\
&+S_{3y}^3S_{4y}^3\big(-S_{4x}^2+S_{4z}^2\big)\big)\big)\\
A_1^{3,4,4} := \frac{1}{12\sqrt{14}}&\big(9S_{2a}^3\big(S_{3y}^4S_{4y}^2\big(S_{4x}^2-S_{4z}^2\big)+S_{3x}^4S_{4x}^2\big(S_{4y}^2-S_{4z}^2\big)\big)\\
&+9S_{2a}S_{2b}^2\big(S_{3y}^4S_{4y}^2\big(S_{4x}^2-S_{4z}^2\big)+S_{3x}^4S_{4x}^2\big(S_{4y}^2-S_{4z}^2\big)\big)\\
&+9\sqrt{3}S_{2a}^2S_{2b}\big(S_{3x}^4S_{4x}^2\big(S_{4y}^2-S_{4z}^2\big)+S_{3y}^4S_{4y}^2\big(-S_{4x}^2+S_{4z}^2\big)\big)\\
&+\sqrt{3}S_{2b}^3\big(8S_{3z}^4\big(-S_{4x}^2+S_{4y}^2\big)S_{4z}^2+S_{3x}^4S_{4x}^2\big(S_{4y}^2-S_{4z}^2\big)\\
&+S_{3y}^4S_{4y}^2\big(-S_{4x}^2+S_{4z}^2\big)\big)\big)\\
A_1^{3,5,3} := \frac{1}{12\sqrt{14}}&\big(9S_{2a}^3\big(S_{3x}S_{3y}^4S_{4x}S_{4y}^2-S_{3y}^4S_{3z}S_{4y}^2S_{4z}+S_{3x}^4S_{4x}^2(S_{3y}S_{4y}-S_{3z}S_{4z})\big)\\
&+9S_{2a}S_{2b}^2\big(S_{3x}S_{3y}^4S_{4x}S_{4y}^2-S_{3y}^4S_{3z}S_{4y}^2S_{4z}+S_{3x}^4S_{4x}^2(S_{3y}S_{4y}-S_{3z}S_{4z})\big)\\
&+9\sqrt{3}S_{2a}^2S_{2b}\big(-S_{3x}S_{3y}^4S_{4x}S_{4y}^2+S_{3y}^4S_{3z}S_{4y}^2S_{4z}+S_{3x}^4S_{4x}^2(S_{3y}S_{4y}\\
&-S_{3z}S_{4z})\big)+\sqrt{3}S_{2b}^3\big(S_{3x}^4S_{4x}^2(S_{3y}S_{4y}-S_{3z}S_{4z})\\
&+S_{3y}S_{3z}S_{4y}S_{4z}\big(S_{3y}^3S_{4y}+8S_{3z}^3S_{4z}\big)\\
&-S_{3x}S_{4x}\big(S_{3y}^4S_{4y}^2+8S_{3z}^4S_{4z}^2\big)\big)\big)
\end{aligned}
$$

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$$
\begin{aligned}
A_1^{0,6,6} := \frac{1}{\sqrt{6}}&\big(S_{3y}^4S_{3z}^2S_{4y}^4S_{4z}^2+S_{3y}^2S_{3z}^4S_{4y}^2S_{4z}^4+S_{3x}^4S_{4x}^4\big(S_{3y}^2S_{4y}^2+S_{3z}^2S_{4z}^2\big)\\
&+S_{3x}^2S_{4x}^2\big(S_{3y}^4S_{4y}^4+S_{3z}^4S_{4z}^4\big)\big)\\
A_1^{2,4,6} := \frac{1}{6\sqrt{10}}&\big(6S_{2a}S_{2b}\big(-S_{3y}^2S_{3z}^2S_{4y}^4S_{4z}^2+S_{3x}^2\big(S_{3y}^2S_{4x}^2S_{4y}^2\big(S_{4x}^2-S_{4y}^2\big)\\
&+S_{3z}^2S_{4x}^4S_{4z}^2\big)\big)+3\sqrt{3}S_{2a}^2\big(S_{3y}^2S_{3z}^2S_{4y}^4S_{4z}^2+S_{3x}^2\big(S_{3y}^2S_{4x}^2S_{4y}^2\\
&\big(S_{4x}^2+S_{4y}^2\big)+S_{3z}^2S_{4x}^4S_{4z}^2\big)\big)+\sqrt{3}S_{2b}^2\big(S_{3y}^2S_{3z}^2S_{4y}^2S_{4z}^2\big(S_{4y}^2+4S_{4z}^2\big)\\
&+S_{3x}^2S_{4x}^2\big(S_{3y}^2S_{4y}^2\big(S_{4x}^2+S_{4y}^2\big)+S_{3z}^2S_{4z}^2\big(S_{4x}^2+4S_{4z}^2\big)\big)\big)\big)\\
A_1^{2,5,5} := \frac{1}{6\sqrt{10}}&\big(6S_{2a}S_{2b}\big(-S_{3x}S_{3y}^4S_{4x}S_{4y}^4-S_{3y}^4S_{3z}S_{4y}^4S_{4z}+S_{3x}^4S_{4x}^4(S_{3y}S_{4y}+S_{3z}S_{4z})\big)\\
&+3\sqrt{3}S_{2a}^2\big(S_{3x}S_{3y}^4S_{4x}S_{4y}^4+S_{3y}^4S_{3z}S_{4y}^4S_{4z}\\
&+S_{3x}^4S_{4x}^4(S_{3y}S_{4y}+S_{3z}S_{4z})\big)+\sqrt{3}S_{2b}^2\big(S_{3y}^4S_{3z}S_{4y}^4S_{4z}+4S_{3y}S_{3z}^4S_{4y}S_{4z}^4\\
&+S_{3x}^4S_{4x}^4(S_{3y}S_{4y}+S_{3z}S_{4z})+S_{3x}S_{4x}\big(S_{3y}^4S_{4y}^4+4S_{3z}^4S_{4z}^4\big)\big)\big)
\end{aligned}
$$

$$A_1^{2,6,4} := \frac{1}{6\sqrt{10}}\big(6S_{2a}S_{2b}\big(-S_{3x}^2S_{3y}^4S_{4x}^2S_{4y}^2-S_{3y}^4S_{3z}^2S_{4y}^2S_{4z}^2+S_{3x}^4S_{4x}^2\big(S_{3y}^2S_{4y}^2+S_{3z}^2S_{4z}^2\big)\big)$$
$$+3\sqrt{3}S_{2a}^2\big(S_{3x}^2S_{3y}^4S_{4x}^2S_{4y}^2+S_{3y}^4S_{3z}^2S_{4y}^2S_{4z}^2+S_{3x}^4S_{4x}^2\big(S_{3y}^2S_{4y}^2+S_{3z}^2S_{4z}^2\big)\big)$$
$$+\sqrt{3}S_{2b}^2\big(S_{3y}^2S_{3z}^2\big(S_{3y}^2+4S_{3z}^2\big)S_{4y}^2S_{4z}^2+S_{3x}^4S_{4x}^2\big(S_{3y}^2S_{4y}^2+S_{3z}^2S_{4z}^2\big)$$
$$+S_{3x}^2S_{4x}^2\big(S_{3y}^4S_{4y}^2+4S_{3z}^4S_{4z}^2\big)\big)\big)$$

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